

The de Rham Homotopy Theory and Differential Graded Category

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Abstract

This paper is a generalization of [30]. We develop the de Rham homotopy theory of not necessarily nilpotent spaces. We use two algebraic objects: *closed dg-categories* and *equivariant dg-algebras*. We see these two objects correspond in a certain way (Prop.3.3.4, Thm.3.4.5). We prove an equivalence between the homotopy category of schematic homotopy types [22] and a homotopy category of closed dg-categories (Thm.1.0.1). We give a description of homotopy invariants of spaces in terms of minimal models (Thm.1.0.2). The minimal model in this context behaves much like the Sullivan's minimal model. We also provide some examples. We prove an equivalence between fiberwise rationalizations [5] and closed dg-categories with subsidiary data (Thm.1.0.4).

Keywords: rational homotopy theory, non-simply connected space, dg-category, schematic homotopy type.

1 Introduction

In [8], Sullivan constructed the correspondence between the rational homotopy types of nilpotent spaces and commutative dg-algebras over rationals, which associates polynomial de Rham algebras to spaces. In particular, he showed that homotopy invariants of nilpotent spaces, such as rational homotopy groups, can be derived from the algebras. This Sullivan's theory is called the de Rham homotopy theory.

In the non-nilpotent case, as a generalization of the rationalization, the fiberwise rationalization was proposed by Bousfield and Kan ([5], see also introductions of [16, 30]). For this notion, A.Gómez-Tato, S.Halperin and D.Tanré [16] generalized the Sullivan's result to non-nilpotent spaces.

Recently, as another non-nilpotent generalization of the rationalization, the schematization [22] was introduced by Toën. While the rationalization is the localization with respect to the rational homology groups, the schematization is a candidate for a localization with respect to all cohomology groups with coefficients in finite rank local systems. In this paper, we generalize the Sullivan's result for the schematization over a field of characteristic 0. We use two algebraic objects which are generalizations of commutative dg-algebras: *closed dg-categories* [30] and *equivariant commutative dg-algebras*. These two algebraic objects have different advantages. We establish an equivalence between the homotopy category of schematic homotopy types and a homotopy category of closed dg-categories (Thm.1.0.1). We give a description of homotopy invariants of not necessarily nilpotent spaces in terms of the minimal models of equivariant dg-algebras (Thm.1.0.2).

Let k be a field of characteristic 0. A closed dg-category is a k -linear dg-category which is equipped with a closed tensor structure consistent with the differential graded structure (see Def.2.1.1). A typical

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example of a closed tensor structure is tensors and internal homs on the category of representations of a group. If one views a dg-algebra as a dg-category with only one object, a tensor structure on a dg-category is a natural generalization of commutativity of a dg-algebra. We also need to consider internal homs. The existence of a model category structure on the category of small closed dg-categories was proved in [30].

In [30], a closed dg-category $T_{PL}(K)$ is defined for a simplicial set (or triangulated space) K . Its objects are finite rank k -linear local systems on K and complexes of morphisms are polynomial de Rham complexes with local coefficients. This is an analogue of the dg-category of flat bundles on a manifold, which was defined by Simpson [12, section 3]. We consider $T_{PL}(K)$ as a generalization of the polynomial de Rham algebra which contains information of finite dimensional representation of the fundamental group. As we see in [30], when the fundamental group of K is a finite group and $k = \mathbb{Q}$, the construction $K \mapsto T_{PL}(K)$ is equivalent to the fiberwise rationalization of K . But in general, it is different as we consider only *finite rank* local systems.

We introduce a special class of closed dg-categories which we call Tannakian dg-categories (Def.3.1.1). It is characterized by conditions abstracted from the closed dg-category $T_{PL}(K)$ of connected K . These conditions are stated in terms of Tannakian theory [11] and the completeness of a dg-category [12, section 3]. Tannakian theory concerns a duality between affine group schemes and certain closed tensor k -linear categories (see Appendix A.2). The completeness of the dg-category means that information of exact sequences in the category of 0-th cocycles determines and is determined by the first cohomology groups of morphisms (see subsection 2.2). Tannakian dg-categories are exactly those which correspond to schematic homotopy types.

While closed dg-categories have good functorial and homotopical properties, they are not suitable for computations. To cover this inconvenience, we use a π_1 -equivariant commutative dg-algebra $A_{red}(K)$ defined as follows ($\pi_1 = \pi_1(K)$). Let π_1^{red} be the pro-reductive completion of π_1 over k (see subsection 1.1) and $\mathcal{O}(\pi_1^{red})$ be its coordinate ring. It has two actions of π_1 : the left and right translations. With the right translation, we regard it as a local system on K .

1. As a complex, $A_{red}(K)$ is the polynomial de Rham complex with coefficients in the local system $\mathcal{O}(\pi_1^{red})$.
2. The multiplication is defined from those of polynomial forms and the coordinate ring.
3. The action of π_1 is defined from the left translation on $\mathcal{O}(\pi_1^{red})$.

The importance of this dg-algebra first seemed to be recognized by Deligne and it has been studied by Katzarkov, Pantev, Toën [22, 24, 25] and Pridham [27, 28, 29]. We will prove a correspondence between Tannakian dg-categories and equivariant dg-algebras, where $T_{PL}(K)$ corresponds to $A_{red}(K)$ (see Prop.3.3.4, Thm.3.4.5). The proof is not difficult but this is very useful. For example, we cannot find the natural representations of the fundamental group on higher homotopy groups if we see only the dg-algebra, but these representations appear as objects of the corresponding dg-category. Results in the following are obtained by using this correspondence.

1.0.1 Main results

We shall state main results. Let SHT_* be the category of pointed schematic homotopy types (Def.5.1.1). It is a full subcategory of the category of ∞ -stacks over k , which is characterized by a certain k -linearity of homotopy sheaves. The schematization $(K \otimes k)^{sch}$ of a simplicial set K is a universal schematic homotopy type for K . Let Tan_* be the category of Tannakian dg-categories with a fiber functor. A fiber functor of a Tannakian dg-category C is a dg-functor from C to the category of finite dimensional vector spaces, which preserves closed tensor structures (Def.2.1.3). Let $sSet_*^c$ denote the category of pointed connected simplicial sets.

Theorem 1.0.1 (Thm.5.2.1). *There exists an equivalence of categories $\text{Ho}(\text{SHT}_*) \rightarrow \text{Ho}(\text{Tan}_*)^{\text{op}}$ such that the following diagram is commutative up to natural isomorphisms:*

$$\begin{array}{ccc} \text{Ho}(\text{SHT}_*) & \xrightarrow{\sim} & \text{Ho}(\text{Tan}_*)^{\text{op}} \\ (-\otimes k)^{\text{sch}} \uparrow & & \nearrow \text{T}_{\text{PL}} \\ \text{Ho}(\text{sSet}_*^c). & & \end{array}$$

Here, $\text{Ho}(-)$ denotes the corresponding homotopy category (see subsection 1.1). The equivalence is induced by a Quillen pair between larger model categories.

It is known that the schematization comes from $A_{\text{red}}(K)$. In fact, it is realized as the homotopy quotient of topological realization of $A_{\text{red}}(K)$ by π_1^{red} (see [25, Cor.3.3]). But this construction is not natural with respect to maps between simplicial sets as the construction $K \mapsto \pi_1(K)^{\text{red}}$ is not natural with respect to those which do not preserve semi-simple representations of fundamental groups (see Rem.3.3.2, the naturality part of the statement of [25, Cor.3.3] is wrong. the naturality holds only for the subcategory whose morphisms are those which preserve semi-simple ones). The theorem says the schematization and the construction $K \mapsto \text{T}_{\text{PL}}(K)$ are naturally equivalent. We prove this using the model structure on the category of closed dg-categories and results of [22]. We deduce a similar equivalence in the unpointed case from Thm.1.0.1, see Cor.5.2.7.

A feature of the Sullivan's theory is that the minimal model describes the rational homotopy theory of a space in a very transparent way (see [8, P38, Analogy to topology] or [10, Thm.11.5]). For a simply connected space, the indecomposable modules are dual to the rational homotopy groups and the differentials correspond to rational k-invariants of the Postnikov tower.

The correspondence between dg-categories and dg-algebras enables us to translate homotopy invariants of a space into invariants of minimal algebras and we obtain an analogous description in the non-nilpotent case, as follows. $A_{\text{red}}(K)$ has a minimal model (in the usual sense) with a semi-simple π_1 -action. Let \mathcal{M} be such a minimal model of $A_{\text{red}}(K)$. Let

$$V^1, V^2, \dots, V^i, \dots, \quad V^i \subset \mathcal{M}^i$$

be a sequence of semisimple π_1 -modules generating \mathcal{M} freely as a commutative graded algebra. Of course, V^i is isomorphic to the i -th indecomposable module. As \mathcal{M} is minimal, $d(V^i) \subset \mathcal{M}^1 \otimes_k V^i \oplus \mathcal{M}(i-1)^{i+1}$, where d is the differential of \mathcal{M} and $\mathcal{M}(i-1)$ is the dg-subalgebra of \mathcal{M} generated by $\bigoplus_{j \leq i-1} M^j$. So there is a unique decomposition

$$d|_{V^i} = d^{\mathcal{M}^1 \otimes_k V^i} \oplus d^{\mathcal{M}(i-1)}$$

consisting of two homomorphisms of π_1 -modules $d^{\mathcal{M}^1 \otimes_k V^i} : V^i \rightarrow \mathcal{M}^1 \otimes_k V^i$ and $d^{\mathcal{M}(i-1)} : V^i \rightarrow \mathcal{M}(i-1)^{i+1}$.

Theorem 1.0.2 (Lem.3.3.9, Thm.4.1.3). *We use the above notations. Put $\pi_i = \pi_i(K)$ for $i \geq 1$. For $i \geq 2$, we consider π_i as a π_1 -module by the canonical action. Let $n \geq 2$. Suppose π_1 is algebraically good (Def.4.1.1) and π_i is of finite rank as an Abelian group for each $2 \leq i \leq n$. Then, for each $2 \leq i \leq n$,*

1. the dual $(V^i)^\vee$ is isomorphic to $(\pi_i \otimes_{\mathbb{Z}} k)^{\text{ss}}$, the semisimplification of π_1 -module $\pi_i \otimes_{\mathbb{Z}} k$,
2. the map $d^{\mathcal{M}^1 \otimes_k V^i}$ encodes the information of a presentation of $\pi_i \otimes_{\mathbb{Z}} k$ as successive extensions of irreducible components of $(\pi_i \otimes_{\mathbb{Z}} k)^{\text{ss}}$, and
3. the map $d^{\mathcal{M}(i-1)}$ corresponds to the k -invariant (tensored with k) of the i -th level of the Postnikov tower of K .

In particular, the two data V^i and $d^{\mathcal{M}^1 \otimes V^i}$ determine the π_1 -module $\pi_i \otimes_{\mathbb{Z}} k$ up to isomorphisms.

As examples of algebraically good groups, finitely generated free groups, finitely generated Abelian groups, fundamental groups of Riemann surfaces are known (see [25, subsection 4.3]). For other examples, see [25, Thm.4.16] and Thm.4.2.8. We prove Thm.1.0.2 by an argument similar to the proof of Hirsch Lemma [10, Thm.11.1].

We can also recover cohomology groups of any finite rank k -local coefficients easily from the minimal model. For a semisimple local system or equivalently, a semi-simple representation V of π_1 , it is $H^*((\mathcal{M} \otimes V)^{\pi_1})$, the cohomology group of the complex of invariants of $\mathcal{M} \otimes V$. For a general finite dimensional representation V , we need to twist the complex $(\mathcal{M} \otimes V^{\text{ss}})^{\pi_1}$ by some Maurer-Cartan element before we take cohomology (V^{ss} denotes the semi-simplification of V).

1.0.2 Examples

We shall provide a simple example deduced from Thm.1.0.2.

Example 1.0.3 (Example 4.2.5). *Suppose k is algebraically closed. Let $n \geq 2$ be an integer and M be a finite rank abelian group with \mathbb{Z} -action. Let $K = K(\mathbb{Z}, M, n) := K(\mathbb{Z} \ltimes K(M, n - 1), 1)$. Here, $K(M, n - 1)$ is an Eilenberg Maclane space realized as simplicial abelian group with the induced action of \mathbb{Z} . Let $g \in \text{GL}(M \otimes_{\mathbb{Z}} k)$ be the action of the generator 1 of \mathbb{Z} and $g = g^s + g^n$ be a Jordan decomposition, where g^s is semi-simple and g^n is nilpotent. Then,*

$$V^i = \begin{cases} k \cdot s & (i = 1) \\ ((M \otimes_{\mathbb{Z}} k)^{\text{ss}})^{\vee} & (i = n) \\ 0 & (\text{otherwise}) \end{cases}$$

and $d(s) = 0$, $d(x) = {}^t g^n(x) \cdot s$ for $x \in V^n$. Here, $(M \otimes_{\mathbb{Z}} k)^{\text{ss}}$ is the vector space $M \otimes_{\mathbb{Z}} k$ on which $1 \in \mathbb{Z}$ acts by the semisimple part g^s , and \mathbb{Z} acts on V^1 trivially.

We give an explicit description of the model of components of a free loop space (Prop.4.2.9) under the assumption that the component contains loops which represent an element of the center of the fundamental group. We also present a model of cell attachment (Example 4.2.6). Models of these topological constructions are not known for the formulation of [16]. By a method similar to the proof of Thm.1.0.2, we will prove that for a nilpotent simplicial set K of finite type, the minimal model of $A_{\text{red}}(K)$ is isomorphic to the Sullivan's minimal model with trivial π_1 -action (Thm.4.2.3). We also provide a description of minimal model of a classifying space of a group which is an extension of given group by an abelian group (Thm.4.2.8).

1.0.3 Equivalence with the fiberwise rationalization

In general, it is impossible to recover the fundamental group from the closed dg-category. The best thing we can obtain is the pro-algebraic completion ([2], see also subsection 1.1). So we cannot expect the closed dg-category corresponds to the fiberwise rationalization of a space. Instead, we prove an equivalence between the fiberwise rationalization and the closed dg-category with subsidiary data. We say a pointed connected simplicial set is algebraically good if its fundamental group is algebraically good (see Def.4.1.1) and each higher homotopy groups are finite dimensional \mathbb{Q} -vector spaces. Let $\text{Rep}(\Gamma)$ denotes the category of finite dimensional k -linear representation of a discrete group Γ .

Theorem 1.0.4 (Thm.4.3.4). *There exists a category Tan_*^{+gd} whose objects are triples (T, Γ, ϕ) consisting of an object $T \in \text{Tan}_*$, an algebraically good group Γ , and an equivalence $\phi : Z^0 T \rightarrow \text{Rep}(\Gamma)$ of closed k -categories with a fiber functor. To a simplicial set K whose fundamental group is algebraically good and*

whose higher homotopy groups are of finite rank, we can assign an object $(T_{PL}(K), \pi_1(K), \phi_K) \in \text{Tan}_*^{+gd}$. This construction induces an equivalence between the homotopy category of algebraically good spaces and the homotopy category of $(\text{Tan}_*^{+gd})^{\text{op}}$. Under this equivalence $(T_{PL}(K), \pi_1(K), \phi_K)$ corresponds to the fiberwise rationalization of K .

1.0.4 Relation with other works

We shall mention the relation with other works: [22, 25] and [27]. The schematic homotopy types and schematization are defined over any field of any characteristic. We use results of [22, 25] in this paper. In [27], another kind of algebraic models of spaces is proposed. They are called pro-algebraic homotopy types and realized as certain simplicial affine group schemes. Schematic homotopy types and pro-algebraic homotopy types are closely related. A pro-algebraic homotopy type is considered as a relative pro-unipotent completion of a schematic homotopy type and these two objects are equivalent on those which come from spaces (see [27, Cor.3.57]). Some of the results of this paper were proved earlier by Toën and Pridham (see [22, 25] and [27]). For example, in the notation of Thm.1.0.2, it was proved in [27] that $(V^i)^\vee$ is isomorphic to $\pi_i \otimes k$ as vector spaces under a bit stronger assumption, see [27, Thm.1.58, Rem.4.43]. A feature of our approach is that the closed dg-categories are nearer to the equivariant dg-algebras than other models. In fact, the correspondence between dg-categories and dg-algebras is very clear and so we can obtain descriptions of the action of fundamental group on homotopy groups and k-invariants as in Thm.1.0.2 and prove fundamental theorems 4.2.3 and 4.2.8. These are new results.

1.0.5 Organization of the paper

In section 2, we mainly gather definitions and results from other papers. We recall basic properties of the completion of dg-categories from [12]. We see that the completion, slightly modified, fits in the context of closed dg-categories.

In section 3, we prepare some technical results to prove results in sections 4 and 5. In 3.1 we introduce the notion of Tannakian dg-categories. In 3.3, we compare Tannakian dg-categories and equivariant dg-algebras and deduce some lemmas. In the statements and proofs of these lemmas, we use the model category structure on the category of closed dg-categories (see Thm.2.1.5). We also use internal homs. We introduce a notion of iterated Hirsch extensions of dg-algebras. In 3.4, we define the functor T_{PL} , which we call the generalized de Rham functor, and prove that $T_{PL}(K)$ comes from $A_{\text{red}}(K)$. All arguments in this section are elementary except for the languages of model categories.

In section 4, we recall the notion of algebraically goodness of a discrete group and prove Thm.1.0.2 and 1.0.4. We also provide some examples. For the proof of Thm.1.0.2, we mainly follow the method of [10] and justify a technical part by using the model category structure on cubical sets in [20], see Appendix A.1. In the proofs of some results in this section, we use results of the next section 5.

In section 5, we prove Thm.1.0.1. In 5.1, we recall the notions of schematic homotopy types and schematization from [22] and define a functor from the category of simplicial presheaves to the category of closed dg-categories. This is an analogue of the generalized de Rham functor (and denoted by T_{PL} , too). As for logical order, section 5 is previous to section 4.

In Appendix, we show some variants of the polynomial de Rham theorem and summarize the Tannakian theory of [11].

1.1 Notations and terminologies

We fix a field k of characteristic 0. \mathbb{Q} denotes the field of rational numbers. All differential graded objects are assumed to be defined over k and *non-negatively cohomologically graded*. We denote by $C^{\geq 0}(k)$ the

category of non-negatively cohomologically graded complexes and chain maps. By dg-algebra, we mean commutative dg-algebra (with non-negative cohomological grading). Dg-algebra and dg-category are abbreviated to dga and dgc, respectively. We denote by Vect the category of finite dimensional k -vector spaces and k -linear maps. More precisely, it denotes a suitably small full subcategory, see the paragraph which precedes Def.2.1.3 for definition. We use the notations and terminologies in [30, 1.1]. In particular, for a category (resp. a dg-category) C , $\text{Ob}(C)$ denotes the set of objects of C and $\text{Hom}_C(c_0, c_1)$ denotes the set of morphisms (resp. the complex of morphisms) between two objects c_0 and $c_1 \in \text{Ob}(C)$. $\text{dgCat}^{\geq 0}$ denotes the category of small non-negatively cohomologically graded dg-categories and dg-functors. For a dg-category C , $Z^0 C$ denotes the category of 0-th cocycles of C . Its objects are the same as objects of C , and its morphisms are morphisms of C which are cocycles of degree 0. For two dgc's C and D , $C \boxtimes D$ denotes a dgc whose objects are pairs (c, d) of $c \in \text{Ob}(C)$ and $d \in \text{Ob}(D)$, and whose complexes of morphisms are tensors of those of C and D (see [30]). If a dga A is free as a graded commutative algebra, i.e., it is the tensor of the symmetric algebra generated by even degree generators and the wedge algebra of odd degree ones, we sometimes write $A = \bigwedge(V^i)$ where V^i denotes a module of generators of degree i .

All schemes are assumed to be defined over k . For an affine scheme X , we denote by $\mathcal{O}(X)$ the coordinate ring of X . We always identify a finite dimensional k -vector space with an affine additive group scheme in the obvious way.

Let Γ be a discrete group and G be an affine group scheme. The term, Γ -module or Γ -representation represents the same thing. We always identify G -modules (or G -representations) with $\mathcal{O}(G)$ -comodules (see [11, P.126]). $\text{Rep}^\infty(\Gamma)$ (resp. $\text{Rep}^\infty(G)$) denotes the category of possibly infinite dimensional Γ -modules (resp. G -modules) over k and $\text{Rep}(\Gamma)$ (resp. $\text{Rep}(G)$) denotes the full subcategory of $\text{Rep}^\infty(\Gamma)$ (resp. $\text{Rep}^\infty(G)$) consisting of finite dimensional objects. For a technical reason, we need to make the categories $\text{Rep}(\Gamma)$ and $\text{Rep}(G)$ suitably small. See the paragraph which precedes Def.2.1.3 for the precise definition. $\mathcal{O}(G)$ has two natural G -action: the right and left translations. We denote by $\mathcal{O}(G)_r$ (resp. $\mathcal{O}(G)_l$) $\mathcal{O}(G)$ considered as a G -module by the right (resp. left) translation. We say a representation (of a discrete group or a group scheme) is semisimple if it can be decomposed into a direct sum of irreducible representations. We say an affine group scheme is *pro-reductive* (or simply, *reductive*) if any of its representations is semi-simple. Any affine group scheme G has a decomposition:

$$G \cong R_u(G) \rtimes G^{\text{red}},$$

where $R_u(G)$ is the pro-unipotent radical of G , and $G^{\text{red}} = G/R_u(G)$ is pro-reductive (see [26] for details). Representations of G^{red} are in one to one correspondence with semi-simple representations of G via the pullback by the projection $G \rightarrow G^{\text{red}}$.

The *pro-algebraic completion* of Γ ([2]), denoted by Γ^{alg} , is an affine group scheme over k with a group homomorphism $\psi_\Gamma : \Gamma \rightarrow \Gamma^{\text{alg}}(k)$, where $\Gamma^{\text{alg}}(k)$ denotes the group of k -valued points of Γ^{alg} , such that finite dimensional k -linear representations of Γ are in one to one correspondence with finite dimensional representations of Γ^{alg} via the pullback of action by ψ_Γ . We put $\Gamma^{\text{red}} := (\Gamma^{\text{alg}})^{\text{red}}$ and call it the *pro-reductive completion* of Γ . Finite dimensional representation of Γ^{red} are in one to one correspondence with finite dimensional semi-simple k -representation of Γ via the pullback by $\Gamma \xrightarrow{\psi_\Gamma} \Gamma^{\text{alg}}(k) \rightarrow \Gamma^{\text{red}}(k)$.

Our notion of model categories is that of [14]. $\text{Ho}(M)$ denotes the homotopy category of a model category M . If M' is a full subcategory of M which is stable under weak equivalences, $\text{Ho}(M')$ denotes the full subcategory of $\text{Ho}(M)$ spanned by M' . This is isomorphic to the localization of M' by weak equivalences. $[-, -]_{M'}$ denotes the set of morphisms of $\text{Ho}(M')$. $s\text{Set}$ (resp. $s\text{Set}_*$) denote the category of simplicial sets (resp. pointed simplicial sets). For a group Γ , $B\Gamma$ or $K(\Gamma, 1)$ denotes the simplicial nerve of Γ .

2 Preliminaries

2.1 Closed dg-categories

The following is a rewrite of [30, Def.2.1.1], where we call the same objects closed tensor dg-categories.

Definition 2.1.1 (closed dg-categories, dgCat^{cl} , Cat^{cl}).

(1) Let C be an object of $\text{dgCat}^{\geq 0}$. A closed tensor structure on C is a 11-tuple

$$((- \otimes -), \mathbf{1}, a, \tau, u, \mathfrak{Hom}, \phi, (- \oplus -), s_1, s_2, \mathbf{0})$$

consisting of

1. a morphism $(-\otimes-): C \boxtimes C \longrightarrow C \in \text{dgCat}^{\geq 0}$,
2. a distinguished object $\mathbf{1} \in C$,
3. natural isomorphisms

$$\begin{aligned} a : ((-\otimes-)\otimes-) &\Longrightarrow (-\otimes(-\otimes-)) : (C \boxtimes C) \boxtimes C \cong C \boxtimes (C \boxtimes C) \longrightarrow C, \\ \tau : (-\otimes-) &\Longrightarrow (-\otimes-) \circ \mathcal{T}_{C,C} : C \boxtimes C \longrightarrow C, \\ u : (-\otimes\mathbf{1}) &\Longrightarrow id_C : C \longrightarrow C \end{aligned}$$

satisfying usual coherence conditions on associativity, commutativity and unity, see [3, pp.251],

4. a morphism $\mathfrak{Hom} : C^{\text{op}} \boxtimes C \longrightarrow C \in \text{dgCat}^{\geq 0}$,

5. a natural isomorphism

$$\phi : \text{Hom}_C(-\otimes-, -) \Longrightarrow \text{Hom}_C(-, \mathfrak{Hom}(-, -)) : C^{\text{op}} \boxtimes C^{\text{op}} \boxtimes C \longrightarrow \mathbb{C}^{\geq 0}(k),$$

6. a morphism $(-\oplus-) : C \times C \longrightarrow C \in \text{dgCat}^{\geq 0}$,

7. two natural transformations

$$P_1 \xrightarrow{s_1} (-\oplus-) \xleftarrow{s_2} P_2 : C \times C \longrightarrow C,$$

where $P_i : C \times C \longrightarrow C$ is the i -th projection, such that the induced morphism $s_1^* \times s_2^* : \text{Hom}_C(c_0 \oplus c_1, c') \longrightarrow \text{Hom}_C(c_0, c') \times \text{Hom}_C(c_1, c')$ is an isomorphism (i.e., $c_0 \oplus c_1$ is a coproduct), and

8. a distinguished object $\mathbf{0} \in \text{Ob}(C)$ such that $\text{Hom}_C(\mathbf{0}, c) = 0$ for any $c \in \text{Ob}(C)$.

We call $(-\otimes-)$ a tensor functor and \mathfrak{Hom} a internal hom functor.

(2) A closed dg-category is an object C of $\text{dgCat}^{\geq 0}$ equipped with a closed tensor structure. For two closed dg-categories C, D , a morphism of closed dg-categories is a morphism $F : C \rightarrow D$ of dgc's which preserves all of the above structures. For example, $F(c \otimes d) = F(c) \otimes F(d)$ (not only naturally isomorphic), $F(\tau_{c,c'}) = \tau_{Fc,Fc'}$ and $F(\mathbf{1}) = \mathbf{1}$. We denote by dgCat^{cl} the category of small closed dgc's.

(3) A closed k -category is a closed dg-category whose complexes of morphisms are concentrated in degree 0 and a morphism of closed k -categories is the same as a morphism of closed dg-categories. We denote by Cat^{cl} the category of small closed k -categories.

The important part of this definition is the data concerning $(-\otimes-)$ and \mathfrak{Hom} . We assume the existence of coproducts in order to make the initial object of dgCat^{cl} be (equivalent to) the closed k -category of finite dimensional k -vector spaces

Notation. We set $C(c) := \text{Hom}_C(\mathbf{1}, c)$ for a closed dgc C and an object $c \in \text{Ob}(C)$.

Example 2.1.2. (1) Let Γ be a discrete group. The k -linear category $\text{Rep}(\Gamma)$ of finite dimensional k -linear representations has a closed tensor structure. The tensor of two representations is, as usual, the tensor of vector spaces with the diagonal action and the internal hom is similar.

(2)(The dg-category of flat bundles [12]) Let X be a C^∞ -manifold. A flat bundle (V, D) on X is a pair of a C^∞ -vector bundle V and a flat connection $D : V \rightarrow \mathcal{A}_X^1 \otimes V$, where \mathcal{A}_X^1 is the sheaf of C^∞ one-forms on X . The tensor of two flat bundles $(V, D), (V', D')$ is the pair of the tensor of vector bundles $V \otimes V'$ and the flat connection $D \otimes \text{id} + \text{id} \otimes D'$. The internal hom is similar.

The dg-category \mathcal{C}_{dR} of flat bundles on X is defined as follows. Its objects are flat bundles on X and its complex of morphisms between (V, D) and (V', D') is the twisted de Rham complex of forms with coefficients in the internal hom $\mathfrak{Hom}((V, D), (V', D'))$. It is easy to see the tensor and the internal hom defined above are extended to a closed tensor structure on \mathcal{C}_{dR} . See [12] for details.

(3) The reader may feel the definition of morphism of closed dgc's is non-natural as it does not require natural isomorphisms but equalities. The motivation of this definition is to ensure that dgCat^{cl} is closed under limits and colimits. An example of a morphism of dgCat^{cl} is the functor $\text{Rep}(\Gamma') \rightarrow \text{Rep}(\Gamma)$ induced by a group homomorphism $\Gamma \rightarrow \Gamma'$.

We apply the notions of an equivalence and a quasi-equivalence to morphisms of dgCat^{cl} via the forgetful functor $\text{dgCat}^{\text{cl}} \rightarrow \text{dgCat}^{\geq 0}$. For example, we say a morphism in dgCat^{cl} is an equivalence if it induces an equivalence of underlying categories. Note that an equivalence in dgCat^{cl} does not always have a quasi-inverse which is a morphism of dgCat^{cl} . We say two objects of dgCat^{cl} are equivalent if they can be connected by a finite chain of equivalences in dgCat^{cl} .

We denote the initial object of dgCat^{cl} by Vect . Vect is equivalent to the closed k -category of all finite dimensional k -vector spaces and k -linear maps. In fact, Vect is identified with the smallest full subcategory which includes the distinguished objects $\mathbf{1}$ and $\mathbf{0}$ and is closed under \otimes , \mathfrak{Hom} , and \oplus .

In the rest of the paper, we assume a vector space which underlies a finite dimensional representation of a discrete group or an affine group scheme belongs to Vect .

Definition 2.1.3 (closed dg-categories with a fiber functor, $\text{dgCat}_*^{\text{cl}}, \text{Cat}_*^{\text{cl}}$).

(1) The category of closed dg-categories with a fiber functor is the over category

$$\text{dgCat}^{\text{cl}}/\text{Vect}$$

and denoted by $\text{dgCat}_*^{\text{cl}}$. An object (C, ω_C) of $\text{dgCat}_*^{\text{cl}}$ consists of a closed dg-category C and a morphism $\omega_C : C \rightarrow \text{Vect} \in \text{dgCat}^{\text{cl}}$. We call ω_C the fiber functor of C .

(2) A closed k -category with a fiber functor is a closed dg-category with a fiber functor whose complexes of morphisms are concentrated in degree 0 and morphisms of closed k -categories with a fiber functor are the same as morphisms of closed tensor dg-categories with a fiber functor. We denote by Cat_*^{cl} the category of small closed k -categories with a fiber functor.

Example 2.1.4. Let Γ (resp. G) be a discrete group (resp. an affine group scheme). We regard $\text{Rep}(\Gamma)$ (resp. $\text{Rep}(G)$) as an object of Cat_*^{cl} with the forgetful functor to Vect .

The following is proved in [30].

Theorem 2.1.5 (Thm.2.3.2 of [30]). (1) The category dgCat^{cl} has a cofibrantly generated model category structure where weak equivalences and fibrations are defined as follows.

- A morphism $F : C \rightarrow D \in \text{dgCat}^{\text{cl}}$ is a weak equivalence if and only if it is a quasi-equivalence.
- A morphism $F : C \rightarrow D \in \text{dgCat}^{\text{cl}}$ is a fibration if and only if it satisfies the following two conditions.
 - For $c, c' \in \text{Ob}(C)$ the morphism $F_{(c,c')} : \text{Hom}_C(c, c') \rightarrow \text{Hom}_D(Fc, Fc')$ is a levelwise epimorphism.
 - For any $c \in \text{Ob}(C)$ and any isomorphism $f : Fc \rightarrow d' \in Z^0(D)$, there exists an isomorphism $g : c \rightarrow c' \in Z^0(C)$ such that $F(g) = f$.

(2) $\text{dgCat}_*^{\text{cl}}$ has a model category structure induced by that of dgCat^{cl} .

2.2 Completeness of dg-category

We shall recall the notion of completeness of a dg-category from [12, section 3]. Let C be a dg-category. An extension in C is a pair of morphisms

$$c_0 \xrightarrow{a} c_2 \xrightarrow{b} c_1$$

with $a, b \in \text{Hom}^0$, $b \circ a = 0$ and $d(a) = 0$, $d(b) = 0$, such that a splitting exists: a splitting is a pair of morphisms of degree 0

$$c_0 \xleftarrow{g} c_2 \xleftarrow{h} c_1$$

such that $ga = id_{c_0}$, $bh = id_{c_1}$ and $ag + hb = id_{c_2}$. We define a morphism $\delta \in \text{Hom}^1(c_1, c_0)$ by $\delta = gd(h)$. $d(\delta) = 0$ and δ defines a class $[\delta] \in H^1(\text{Hom}(c_1, c_0))$. It is easy to check that this class is independent of a choice of splittings. We call $[\delta]$ the class of the extension $c_0 \rightarrow c_2 \rightarrow c_1$.

Definition 2.2.1. We say a dg-category C is complete if for each $c_0, c_1 \in \text{Ob}(C)$ any class in $H^1(\text{Hom}(c_1, c_0))$ is a class of some extension.

Example 2.2.2. Let \mathcal{C}_{dR} be the category of flat bundles defined in Example 2.1.2,(2). As in [12], \mathcal{C}_{dR} is complete. In fact, for a cocycle $\delta \in \text{Hom}_{\mathcal{C}_{dR}}^1((V, D), (V', D'))$, we define a flat bundle (V'', D'') by $V'' = V \oplus V'$ and $D'' = \begin{pmatrix} D & 0 \\ \delta & D' \end{pmatrix}$. The sequence $(V', D') \rightarrow (V'', D'') \rightarrow (V, D)$ is an extension corresponding to δ .

We will construct the completion of a dgc C which has finite coproducts. We define a dgc \overline{C} as follows. The objects of \overline{C} are pairs (c, η) with $c \in \text{Ob}(C)$ and $\eta \in \text{Hom}^1(c, c)$ such that

$$d(\eta) + \eta^2 = 0.$$

(We call an element η satisfying the above equation a *Maurer-Cartan (MC) element on c* .) We set

$$\text{Hom}^n((c, \eta), (c', \eta')) := \text{Hom}^n(c, c')$$

and

$$d_{\overline{C}}(f) := d_C(f) + \eta' \circ f - (-1)^{\deg f} f \circ \eta.$$

We identify C with the full sub-dg-category of \overline{C} consisting of objects of the form $(c, 0)$, $c \in C$ and define the completion \widehat{C} to be the smallest full sub-dg-category of \overline{C} including C and closed under extensions (and isomorphisms).

The following is a rewrite of [12, Lemma 3.1, Lemma 3.3].

Lemma 2.2.3 ([12]). (1) Let C be a dgc closed under finite coproducts. \widehat{C} is complete. For two objects $(c_0, \eta_0), (c_1, \eta_1) \in \widehat{C}$, a MC-element corresponding to a cocycle $\alpha \in Z^1(\text{Hom}_{\widehat{C}}((c_1, \eta_1), (c_0, \eta_2)))$ is given by $\begin{pmatrix} \eta_0 & \alpha \\ 0 & \eta_1 \end{pmatrix}$.

(2) The functor $C \rightarrow \widehat{C}$, $c \mapsto (c, 0)$ is initial up to natural isomorphisms among dg-functors $C \rightarrow D$ with D complete. More precisely, for such functor $F : C \rightarrow D$ there exists a functor $\widehat{F} : \widehat{C} \rightarrow D$ factorizing F and such \widehat{F} is unique up to unique natural isomorphisms.

(3) Let $F : C \rightarrow D$ be a quasi-equivalence. The induced functor $\widehat{F} : \widehat{C} \rightarrow \widehat{D}$ is also a quasi-equivalence.

2.2.1 Completion and closed dg-categories

A closed tensor structure on C induces a closed tensor structure on \widehat{C} as follows.

$$(c_1, \eta_1) \otimes (c_2, \eta_2) = (c_1 \otimes c_2, \eta_1 \otimes \text{id} + \text{id} \otimes \eta_2),$$

$$\mathfrak{Hom}((c_1, \eta_1), (c_2, \eta_2)) = (\mathfrak{Hom}(c_1, c_2), \mathfrak{Hom}(\text{id}, \eta_2) - \mathfrak{Hom}(\eta_1, \text{id})).$$

The other structures such as homomorphisms between complexes of morphisms and natural isomorphisms are the same as those of C . If C is a closed dgc, we consider \widehat{C} as a closed dgc with this induced structure. Note that this closed dgc does not have a universality like Lem.2.2.3 in dgCat^{cl} . We shall modify \widehat{C} . The modification makes the tensor and the internal hom "free". For example, we want to avoid two objects $(c_0 \otimes c_1) \otimes c_2$ and $c_0 \otimes (c_1 \otimes c_2)$ happen to be equal.

Let D be a closed dgc. Let $W_{\text{cl}}(\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\})$ denote the set of the words generated by $\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}$ with operations \otimes' , \mathfrak{Hom}' and \oplus' ($\mathbf{1}$ and $\mathbf{0}$ are formal symbols, see [30, sub-subsection 2.2.2] for an explicit definition). We regard $\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}$ as a subset of $W_{\text{cl}}(\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\})$. Let

$$R : W_{\text{cl}}(\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}) \longrightarrow \text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}$$

be the function given by

1. $RX = X$ for $X \in \text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}$.
2. $R(X \otimes' Y) = RX \otimes RY$, $R(\mathfrak{Hom}'(X, Y)) = \mathfrak{Hom}(RX, RY)$ and $R(X \oplus' Y) = RX \oplus RY$, inductively.

We define a closed dgc D^c as a "pullback" of D by R :

$$\text{Ob}(D^c) = W_{\text{cl}}(\text{Ob}(D) \sqcup \{\mathbf{1}, \mathbf{0}\}), \quad \text{Hom}_{D^c}(X, Y) = \text{Hom}_D(RX, RY).$$

Clearly the construction

$$\text{dgCat}^{\text{cl}} \ni D \longmapsto D^c \in \text{dgCat}^{\text{cl}}$$

is functorial and R induces a natural equivalence $R_D : D^c \longrightarrow D \in \text{dgCat}^{\text{cl}}$.

Lemma 2.2.4. Let $E \in \text{dgCat}^{\text{cl}}$. Let $C, D \in \text{dgCat}^{\text{cl}}/E$. Suppose D is complete.

(1) Suppose D is fibrant in $\text{dgCat}^{\text{cl}}/E$. We regard C^c as an object over E whose augmentation $a_{C^c} : C^c \rightarrow E$ is the composition $C^c \xrightarrow{R_C} C \longrightarrow E$. Suppose an augmentation $a_{\widehat{C}^c} : (\widehat{C})^c \rightarrow E \in \text{dgCat}^{\text{cl}}$ such that $a_{\widehat{C}^c} \circ (I_C)^c = a_{C^c}$ ($I_C : C \rightarrow \widehat{C}$ is the canonical inclusion) is given. Let $F : C^c \longrightarrow D \in \text{dgCat}^{\text{cl}}/E$ be a morphism. There exists a morphism $\tilde{F} : (\widehat{C})^c \longrightarrow D \in \text{dgCat}^{\text{cl}}/E$ such that the following diagram is commutative.

$$\begin{array}{ccc} C^c & \xrightarrow{F} & D \\ (I_C)^c \downarrow & \nearrow \tilde{F} & \\ \widehat{C}^c & & \end{array}$$

Let $\tilde{F}' : (\widehat{C})^c \rightarrow D \in \text{dgCat}^{\text{cl}}/E$ be another morphism with $F = \tilde{F}' \circ (I_C)^c$. Then there exists a unique natural isomorphism $\varphi : \tilde{F} \Rightarrow \tilde{F}'$ preserving tensors, such that $\varphi|_{C^c}$ and $(a_D)_*(\varphi)$ are the identities ($a_D : D \rightarrow E$ is the augmentation of D).

(2) Assume the inclusion $E \rightarrow \widehat{E}$ is an isomorphism (i.e., MC-elements in E are all zeros). The pullback by the canonical functor $C \rightarrow \widehat{C}$ gives a bijection:

$$[\widehat{C}, D]_{\text{dgCat}^{\text{cl}}/E} \cong [C, D]_{\text{dgCat}^{\text{cl}}/E}.$$

Here, the augmentation $\widehat{C} \rightarrow E$ is given by $\widehat{C} \rightarrow \widehat{E} \cong E$.

Proof. (1) We first consider the case of $E = *$. As in [12], one can choose a functor $F_1 : \widehat{C}^c \rightarrow D$ such that $F = F_1 \circ (I_C)^c$ and F_1 preserves tensors and internal hom's up to natural isomorphisms which are compatible with coherency isomorphisms. Since the objects of \widehat{C}^c are freely generated by objects of \widehat{C} and $\{\mathbf{0}, \mathbf{1}\}$, one can modify F_1 so that it becomes a morphism of dgCat^{cl} . Thus we get \tilde{F} . The latter part is similar to [12]. For general E , we first take a morphism $F_2 : \widehat{C}^c \rightarrow D \in \text{dgCat}^{\text{cl}}$ such that $F = F_2 \circ (I_C)^c$. By the latter claim in the case of $E = *$, we have a unique natural isomorphism $\varphi : a_D \circ F_2 \Rightarrow a_{\widehat{C}^c}$. Let $x \in \text{Ob}(\widehat{C}) \sqcup \{\mathbf{1}, \mathbf{0}\}$. As the augmentation $a_D : D \rightarrow E \in \text{dgCat}^{\text{cl}}$ is a fibration, one can lift φ_x to an isomorphism $f_x : F_2(x) \rightarrow {}^3\tilde{F}(x) \in D$. Using f_x 's, one can modify F_2 so that $a_D \circ F_2 = a_{\widehat{C}^c}$. This modified F_2 is the required \tilde{F} . The latter claim follows from the case of $E = *$.

(2) We may assume C is cofibrant and D is fibrant in $\text{dgCat}^{\text{cl}}/E$. Note that C^c and \widehat{C}^c are cofibrant. Indeed, let $Q(C^c)$ be a cofibrant replacement of C^c with a trivial fibration $Q(C^c) \rightarrow C^c$. As the composition $Q(C^c) \rightarrow C^c \rightarrow C$ is a trivial fibration, we can take a right inverse $C \rightarrow Q(C^c)$. Using this morphism, one can see the map $Q(C^c) \rightarrow C^c$ has a right inverse so C^c is cofibrant. For \widehat{C}^c , one can find a right inverse of a trivial fibration $Q(\widehat{C}^c) \rightarrow \widehat{C}^c$ using (1). We only have to check the map

$$((I_C)^c)^* : \text{Hom}_{\text{dgCat}^{\text{cl}}/E}(\widehat{C}^c, D) \longrightarrow \text{Hom}_{\text{dgCat}^{\text{cl}}/E}(C^c, D)$$

induces a bijection between the sets of right homotopy classes. The surjectivity follows from (1). Let $\tilde{F}_1, \tilde{F}_2 : \widehat{C}^c \rightarrow D \in \text{dgCat}^{\text{cl}}/E$ be two morphisms such that $\tilde{F}_1|_{C^c}$ and $\tilde{F}_2|_{C^c}$ are right homotopic. Let PD be a path object of D . We may regard C^c , \widehat{C}^c and PD as objects of $\text{dgCat}^{\text{cl}}/D \times_E D$. Then by (1), there exists a morphism $\tilde{H} : \widehat{C}^c \rightarrow PD \in \text{dgCat}^{\text{cl}}/E$ such that the following diagram commutes.

$$\begin{array}{ccc} C^c & \xrightarrow{H} & PD \\ \downarrow & \nearrow \tilde{H} & \downarrow \\ \widehat{C}^c & \xrightarrow[\tilde{F}_1 \times \tilde{F}_2]{} & D \times_E D \end{array}$$

Thus, \tilde{F}_1 and \tilde{F}_2 are right homotopic. □

3 Tannakian dg-categories and reductive equivariant dg-algebras

3.1 Tannakian dg-categories

We shall define Tannakian dg-categories. For the definition of neutral Tannakian categories, see Appendix A.2.

Definition 3.1.1 (Tannakian dgc's, Tan , Tan_*). *Let C be a closed dg-category. We say C is a Tannakian dg-category if the following conditions are satisfied.*

- $Z^0(C)$ is a neutral Tannakian category with respect to the closed tensor structure induced from that of C or equivalently, there exist an affine group scheme G over k and a finite chain of morphisms of closed k -categories:

$$Z^0(C) \rightarrow C_1 \leftarrow C_2 \rightarrow \cdots \leftarrow C_n \rightarrow \text{Rep}(G),$$

where all arrows are equivalences of underlying categories. In particular, $Z^0(C)$ is an Abelian category.

- C is complete (see Def.2.2.1).

- If $0 \rightarrow c_0 \rightarrow c_1 \rightarrow c_2 \rightarrow 0$ is a short exact sequence in $Z^0(C)$ then $c_0 \rightarrow c_1 \rightarrow c_2$ is an extension in C in the sense explained in the beginning of subsection 2.2 .

We denote by Tan the full subcategory of dgCat^{cl} consisting of Tannakian dg-categories. We denote by Tan_* the full subcategory of $\text{dgCat}_*^{\text{cl}}$ consisting of objects whose underlying closed dgc belongs to Tan .

The third condition means that extensions in terms of representations and extensions in terms of a dgc coincide. Let C be a Tannakian dg-category and C^{ss} denote full sub dg-category of C consisting of semisimple objects of Z^0C . The third one is equivalent to the one that the morphism $\widehat{C^{\text{ss}}} \rightarrow C$ induced by the natural inclusion is a quasi-equivalence. So Tan and Tan_* are stable under weak equivalences of dgCat^{cl} and $\text{dgCat}_*^{\text{cl}}$, respectively. Note that for a Tannakian dg-category with a fiber functor $(C, \omega_C) \in \text{Tan}_*$, the functor $Z^0\omega_C : Z^0C \rightarrow \text{Vect}$ is automatically exact and faithful.

3.2 Equivariant dg-algebras

Let G be an affine group scheme. We denote by $\text{dgAlg}(G)$ the category of G -equivariant dg-algebras. An object of $\text{dgAlg}(G)$ is a commutative dg-algebra with a $\mathcal{O}(G)$ -comodule structure which is compatible with the grading, the differential and the algebra structure. A morphism of $\text{dgAlg}(G)$ is a morphism of dga's which is compatible with the $\mathcal{O}(G)$ -comodule structures. G -equivariant dg-algebra is abbreviated to G -dga. We say a G -dga A is connected if $H^0A \cong k$. $\text{dgAlg}(G)_0$ denotes the full subcategory of $\text{dgAlg}(G)$ consisting of connected objects.

The following can be proved by an argument similar to the non-equivariant case.

Proposition 3.2.1. *Let G be a reductive affine group scheme. The category $\text{dgAlg}(G)$ admits a model category structure such that*

- a morphism $f : A \rightarrow B \in \text{dgAlg}(G)$ is a weak equivalence if and only if it is a quasi-isomorphism (of underlying complexes), and
- a morphism $f : A \rightarrow B \in \text{dgAlg}(G)$ is a fibration if and only if it is a levelwise epimorphism.

Definition 3.2.2 (minimal G -dga's). *Let G be an affine group scheme. We say a connected G -dga is minimal if its underlying dga is minimal in the usual sense (see [6, 10]). Let A be a connected G -dga. A minimal model of A is a minimal G -dga \mathcal{M} such that there exists a weak equivalence $\mathcal{M} \rightarrow A$ of G -dga's. For a minimal G -dga \mathcal{M} , the G -module*

$$[\mathcal{M}/(\mathcal{M}^{\geq 1} \cdot \mathcal{M}^{\geq 1})]^i$$

is called the i -th indecomposable module of \mathcal{M} . Here, $\mathcal{M}^{\geq 1} \cdot \mathcal{M}^{\geq 1}$ is the submodule of \mathcal{M} generated by $\{x \cdot y \mid \deg x \geq 1, \deg y \geq 1\}$.

The following is well-known.

Proposition 3.2.3. *Let G be a pro-reductive affine group scheme. Then, any connected G -dga has a minimal model. such minimal model is unique up to non-unique natural isomorphisms.*

Proof. The proof is similar to the usual trivial group case. See [6, 10, 8]. \square

Definition 3.2.4 (reductive dga's, $\text{dgAlg}_0^{\text{red}}$, $\text{dgAlg}_{0,*}^{\text{red}}$).

1. A reductive equivariant dg-algebra in short, a reductive dga is a pair (G, A) of a pro-reductive affine group scheme G and a G -dga A .
2. A morphism of reductive dga's $f : (G, A) \rightarrow (H, B)$ is a pair of a morphism of group schemes $f^{\text{gr}} : H \rightarrow G$ and a morphism of H -dga's $f : (f^{\text{gr}})^* A \rightarrow B$. (Note that f^{gr} defines a functor $(f^{\text{gr}})^* : \text{dgAlg}(G) \rightarrow \text{dgAlg}(H)$ by pulling back the group action.)
3. A morphism of reductive dga's $f : (G, A) \rightarrow (H, B)$ is said to be a quasi-isomorphism if f^{gr} is an isomorphism and $f : (f^{\text{gr}})^* A \rightarrow B$ is a quasi-isomorphism.

The category of reductive dga's is denoted by $\text{dgAlg}^{\text{red}}$. We say a reductive dga (G, A) is connected if $H^0(A) \cong k$. We denote by $\text{dgAlg}_0^{\text{red}}$ the full subcategory of $\text{dgAlg}^{\text{red}}$ consisting of connected objects and by $\text{dgAlg}_{0,*}^{\text{red}}$ the over category $\text{dgAlg}_0^{\text{red}}/(e, k)$, where e is the trivial group. We always identify the category $\text{dgAlg}(G)$ with a subcategory of $\text{dgAlg}^{\text{red}}$ in the obvious way. For an object $(G, A) \in \text{dgAlg}_0^{\text{red}}$, a minimal model of (G, A) is a minimal model of A as a G -dga in the sense of Def.3.2.2

3.3 Comparison

In this subsection, we show a correspondence between reductive equivariant dga's and Tannakian dgc's. The direction from dga to dgc is functorial but the other direction is not so and we only have a function $A_{\text{red}} : \text{Ob}(\text{Tan}_*) \rightarrow \text{Ob}(\text{dgAlg}_{0,*}^{\text{red}})$, see Rem.3.3.2.

We shall define two functors

$$\mathsf{T}^{\text{ss}} : \text{dgAlg}_0^{\text{red}} \longrightarrow \text{dgCat}^{\text{cl}}, \quad \mathsf{T} : \text{dgAlg}_0^{\text{red}} \longrightarrow \text{Tan}.$$

Let $A = (G, A) \in \text{dgAlg}_0^{\text{red}}$. For a comodule $M \in \text{Rep}^\infty(G)$, we define the module of invariants M^G by

$$M^G := \text{Ker}(\text{id}_M \otimes k - \rho_M : M \longrightarrow M \otimes \mathcal{O}(G)),$$

where $k : k \rightarrow \mathcal{O}(G)$ is the unit map and $\rho_M : M \rightarrow M \otimes \mathcal{O}(G)$ is the coaction. We set

$$\text{Ob}(\mathsf{T}^{\text{ss}} A) := \text{Ob}(\text{Rep}(G)), \quad \text{Hom}_{\mathsf{T}^{\text{ss}} A}(V, W) := (\mathfrak{Hom}(V, W) \otimes A)^G.$$

Here, \mathfrak{Hom} is the internal hom of $\text{Rep}(G)$, and $\mathfrak{Hom}(V, W) \otimes A$ is considered as a complex of comodules. $(\mathfrak{Hom}(V, W) \otimes A)^G$ is defined by taking invariants in the degreewise manner. We define the composition and closed tensor structure of $\mathsf{T}^{\text{ss}}(A)$ using corresponding structures of $\text{Rep}(G)$ and the multiplication of A . A morphism $f : (G, A) \rightarrow (H, B)$ of $\text{dgAlg}_0^{\text{red}}$ gives a functor $(f^{\text{gr}})^* : \text{Rep}(G) \rightarrow \text{Rep}(H)$ and f induces a morphism $\mathsf{T}^{\text{ss}} f : \mathsf{T}^{\text{ss}} A \rightarrow \mathsf{T}^{\text{ss}} B$ of closed dg-categories. We set

$$\mathsf{T} A := \widehat{\mathsf{T}^{\text{ss}} A}.$$

The right hand side is the completion (see subsection 2.2). As $\mathsf{T}(e, k) \cong \mathsf{T}^{\text{ss}}(e, k) \cong \text{Vect}$, a morphism $(G, A) \rightarrow (e, k)$ defines morphisms $\mathsf{T}^{\text{ss}}(G, A) \rightarrow \text{Vect}$, $\mathsf{T}(G, A) \rightarrow \text{Vect}$ so we obtain functors between augmented categories:

$$\mathsf{T}^{\text{ss}} : \text{dgAlg}_{0,*}^{\text{red}} \longrightarrow \text{dgCat}_*^{\text{cl}}, \quad \mathsf{T} : \text{dgAlg}_{0,*}^{\text{red}} \longrightarrow \text{Tan}_*.$$

We omitt the proof of the following. See [26, Lem.1.3].the reductivity of G is necessary for (2).

Lemma 3.3.1. (1) For $V \in \text{Ob}(\text{Rep}^\infty(G))$ the coaction $V \rightarrow V \otimes_k \mathcal{O}(G)$ induces an isomorphism of vector spaces: $V \rightarrow (V \otimes \mathcal{O}(G)_r)^G$. Similarly, the coaction also induces an isomorphism $V \rightarrow V \otimes^G \mathcal{O}(G)_l$. Here,

$$V \otimes^G \mathcal{O}(G)_l = \text{Ker}((\text{id}_V \otimes \tau) \circ (\rho_V \otimes \text{id}_{\mathcal{O}(G)}) - \text{id}_V \otimes \lambda_G : V \otimes \mathcal{O}(G)_l \rightarrow V \otimes \mathcal{O}(G)_l \otimes \mathcal{O}(G)),$$

where $\tau : \mathcal{O}(G) \otimes \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$ is the isomorphism given by $\tau(x \otimes y) = y \otimes x$ and λ_G is the coaction of the left translation. (In short, $V \otimes^G \mathcal{O}(G)_l = \{\Sigma_i v_i \otimes l_i; \Sigma_i g \cdot v_i \otimes l_i = \Sigma_i v_i \otimes g \cdot l_i \forall g \in G\}$) In particular, as submodules of $V \otimes \mathcal{O}(G)$, $(V \otimes \mathcal{O}(G)_r)^G = V \otimes^G \mathcal{O}(G)_l$.

(2) If $f : A \rightarrow B \in \text{dgAlg}(G)_0$ is a quasi-isomorphism, the induced morphism $\mathsf{T}f : \mathsf{T}A \rightarrow \mathsf{T}B \in \text{dgCat}^{\text{cl}}$ is a quasi-equivalence.

(3) $\mathsf{T}A$ is a Tannakian dgc for any $A \in \text{dgAlg}_0^{\text{red}}$.

Remark 3.3.2. If we define the homotopy category $\text{Ho}(\text{dgAlg}_{0,*}^{\text{red}})$ of $\text{dgAlg}_{0,*}^{\text{red}}$ as the localization of $\text{dgAlg}_{0,*}^{\text{red}}$ obtained by inverting quasi-isomorphisms, by Lemma 3.3.1 T induces a functor between homotopy categories:

$$\mathsf{T} : \text{Ho}(\text{dgAlg}_{0,*}^{\text{red}}) \longrightarrow \text{Ho}(\mathsf{Tan}).$$

Unlike the finite group case [30], this is not an equivalence simply because morphisms between Tannakian dgc's do not always preserve semisimple objects. By the same reason the construction

$$\mathsf{Tan} \ni T \longmapsto A_{\text{red}} T \in \text{dgAlg}_{0,*}^{\text{red}}$$

is not functorial. If we restrict morphisms of Tan_* to those which preserves semi-simple objects, this function is extended to a functor which induces an equivalence of homotopy categories, see Prop.3.3.4 and Lem.3.3.7.

Definition 3.3.3 (the reductive dga associated to T ; $A_{\text{red}} T$).

Let $T = (T, \omega_T)$ be a Tannakian dg-category with a fiber functor. Let $\pi_1(Z^0 T)$ denote the affine group scheme which represents $\underline{\text{Aut}}^\otimes(Z^0 \omega_T)$ (see Appendix A.2). We regard $\mathcal{O}(\pi_1(Z^0 T)^{\text{red}})$ as a $\pi_1(Z^0 T)^{\text{red}}$ -representation by the right translation. We define an augmented $\pi_1(Z^0 T)^{\text{red}}$ -equivariant dg-algebra $A_{\text{red}}(T) \in \text{dgAlg}_{0,*}^{\text{red}}$ as follows. Let

$$\mathcal{O}(\pi_1(Z^0 T)^{\text{red}}) = \bigcup_\lambda V_\lambda$$

be the presentation as the union of finite dimensional $\pi_1(Z^0 T)^{\text{red}}$ -subrepresentations. We regard V_λ 's as objects of T via the equivalence $\widetilde{Z^0 \omega_T} : Z^0 T \rightarrow \text{Rep}(\pi_1(T)^{\text{red}})$ (see Appendix A.2). As a complex, we set

$$A_{\text{red}}(T) = \text{colim}_\lambda T(V_\lambda)$$

(see the notation under Def.2.1.1). The colimit is taken in the category $C^{\geq 0}(k)$ (in this case, this is the set-theoretic union). The k -algebra structure on $\mathcal{O}(\pi_1(Z^0 T)^{\text{red}})$ defines a structure of dga on $A_{\text{red}}(T)$, the left translation of $\pi_1(Z^0 T)^{\text{red}}$ on $\mathcal{O}(\pi_1(Z^0 T)^{\text{red}})$ defines an action on $A_{\text{red}}(T)$ and the counit map of $\mathcal{O}(\pi_1(Z^0 T)^{\text{red}})$ and the fiber functor of T defines an augmentation of $A_{\text{red}}(T)$. Note that we do not say the construction $T \longmapsto A_{\text{red}}(T)$ is functorial.

Proposition 3.3.4. (1) For an augmented 0-connected reductive dga A , $A_{\text{red}} \mathsf{T}(A)$ is isomorphic to A in $\text{dgAlg}_{0,*}^{\text{red}}$.

(2) For a Tannakian dgc (T, ω_T) with a fiber functor, $\mathsf{T} A_{\text{red}}(T)$ is equivalent to T in $\text{dgCat}_*^{\text{cl}}$.

(3) For a Tannakian dgc $T \in \mathsf{Tan}$, there exists a connected reductive dga $A \in \text{dgAlg}_0^{\text{red}}$ such that $\mathsf{T}A$ is equivalent to T as a closed dg-category.

By this proposition, the functor $T : \text{dgAlg}_{0,*}^{\text{red}} \rightarrow \text{Tan}_*$ and the function $A_{\text{red}} : \text{Ob}(\text{Tan}_*) \rightarrow \text{Ob}(\text{dgAlg}_{0,*}^{\text{red}})$ induce a bijection between the isomorphism classes (resp. quasi-isomorphism classes) of dga's and the equivalence classes (resp. quasi-equivalence classes) of closed dgc's.

Proof. (1) This follows from Lem.3.3.1,(1) and the fact that finite limit and filtered colimit (in the category of modules) commutes.

(2) Let $A = A_{\text{red}} T$. We may assume $(Z^0 T)^{\text{ss}} = \text{Rep}(G)$ where $G = \pi_1(Z^0 T)^{\text{red}}$. In the following we deal with $\mathcal{O}(G)_r$ as if it is an object of $\text{Rep}(G)$. As the action of G on $\mathcal{O}(G)_r$ is locally finite, this does not matter.

For an object $V \in \text{Ob}(\text{Rep}(G))$ we have an exact sequence of G -modules:

$$0 \longrightarrow V \xrightarrow{\rho_V} V_u \otimes \mathcal{O}(G)_r \xrightarrow{\phi} V_u \otimes \mathcal{O}(G)_u \otimes \mathcal{O}(G)_r ,$$

where $(-)_u$ denotes the corresponding trivial module and $\phi := (\text{id}_V \otimes \tau) \circ (\rho_V \otimes \text{id}_{\mathcal{O}(G)}) - \text{id}_V \otimes \lambda_G$, see Lem.3.3.1. We have a map between sequences of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & (V \otimes A)^G & \longrightarrow & (V_u \otimes \mathcal{O}(G)_r \otimes A)^G & \xrightarrow{\phi \otimes \text{id}_A} & (V_u \otimes \mathcal{O}(G)_u \otimes \mathcal{O}(G)_r \otimes A)^G \\ & & \downarrow & & \uparrow \rho_A & & \downarrow \rho_A \\ & & V_u \otimes A & \longrightarrow & V_u \otimes \mathcal{O}(G)_u \otimes A & & \\ & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & \text{Hom}_T(\mathbf{1}, V) & \longrightarrow & \text{Hom}_T(\mathbf{1}, V_u \otimes \mathcal{O}(G)_r) & \xrightarrow{\phi_*} & \text{Hom}_T(\mathbf{1}, V_u \otimes \mathcal{O}(G)_u \otimes \mathcal{O}(G)_r). \end{array}$$

As the both horizontal sequences are levelwise exact and the vertical arrows are isomorphisms by Lem.3.3.1, the dotted arrow uniquely exists. We define a dg-functor $F : \text{T}^{\text{ss}} A \longrightarrow T$ by $F(V) = V$ for $V \in \text{Ob}(\text{Rep}(G))$ and $F_{(V,W)} : \text{Hom}_{\text{T}^{\text{ss}} A}(V, W) \longrightarrow \text{Hom}_C(FV, FW)$ being the composition

$$\text{Hom}_{\text{T}^{\text{ss}} A}(V, W) = (\mathfrak{Hom}_{\text{Rep}(G)}(V, W) \otimes A)^G \longrightarrow \text{Hom}_T(\mathbf{1}, \mathfrak{Hom}_T(V, W)) \cong \text{Hom}_T(V, W).$$

One can check this is a morphism of $\text{dgCat}_*^{\text{cl}}$. By definition of Tannakian dgc's, F is extended to an equivalence $F : \text{T} A^c \longrightarrow T \in \text{dgCat}_*^{\text{cl}}$ (see Lem.2.2.4). The proof of (3) is similar to that of (2). \square

The following lemmas are used later

Lemma 3.3.5. (1) Let G be a reductive affine group scheme. Then, $\text{T}^{\text{ss}}(G, k)^c$ is cofibrant in $\text{dgCat}_*^{\text{cl}}$. If $f : A \rightarrow B \in \text{dgAlg}(G)$ be a cofibration between 0-connected G -dga's, then $\text{T}^{\text{ss}}(f)^c : \text{T}^{\text{ss}}(G, A)^c \rightarrow \text{T}^{\text{ss}}(G, B)^c$ is a cofibration in $\text{dgCat}_*^{\text{cl}}$.

(2) Let $(G, A) \in \text{dgAlg}_{0,*}^{\text{red}}$. There exists a commutative square

$$\begin{array}{ccc} \text{T}^{\text{ss}}(G, k)^c & \longrightarrow & \text{Vect} \\ \downarrow & & \downarrow \\ \text{T}^{\text{ss}}(G, A)^c & \longrightarrow & \text{T}^{\text{ss}}(e, A) \end{array}$$

where e denotes the trivial group, the left vertical morphism is induced by the unit $k \rightarrow A$ and the bottom horizontal morphism is induced by the morphism $(G, A) \rightarrow (e, A)$ which is the identity on dg-algebras. This diagram is a pushout square in $\text{dgCat}_*^{\text{cl}}$ and a homotopy pushout square.

Proof. (1) The proof is similar to that of [30, Thm.3.2.10].

(2) We prove the first assertion. Let

$$\begin{array}{ccc} \mathsf{T}^{\text{ss}}(G, k)^c & \xrightarrow{\omega} & \mathbf{Vect} \\ \downarrow & & \downarrow \\ \mathsf{T}^{\text{ss}}(G, A)^c & \xrightarrow{F} & C \end{array}$$

be a commutative square in $\mathsf{dgCat}^{\text{cl}}$. For $V, W \in \text{Ob}(\mathsf{T}^{\text{ss}}(e, A))$ we define a morphism of complex

$$\tilde{F}_{(X, Y)} : \text{Hom}_{\mathsf{T}^{\text{ss}}(e, A)}(V, W) \longrightarrow \text{Hom}_C(V, W)$$

as the following composition.

$$\begin{aligned} \text{Hom}(V, W) \otimes_k A &\cong (\mathfrak{H}\text{om}(V, W) \otimes \mathcal{O}(G) \otimes A)^G \\ &\cong \text{Hom}_{\mathsf{T}^{\text{ss}}A^c}(V, W \otimes \mathcal{O}(G)) \xrightarrow{F} \text{Hom}_C(V, W \otimes F(\mathcal{O}(G))) \xrightarrow{Fu} \text{Hom}_C(V, W). \end{aligned}$$

Here, $u : \mathcal{O}(G) \longrightarrow \mathbf{1} \in \mathbf{Vect}$ denotes the counit map. It is easy to check $F_{(V, W)}$'s form a morphism $\tilde{F} : \mathsf{T}^{\text{ss}}(e, A) \longrightarrow C \in \mathsf{dgCat}_*^{\text{cl}}$ and this is the unique morphism making appropriate diagram commutative. By (1) and [14, Lem.5.2.6], this is a homotopy pushout square so the former one is. \square

Remark 3.3.6. *We can give an explicit cofibrant model for any Tannakian dg-category. In fact, by Lem.2.2.4 and Lem.3.3.5, for a cofibrant connected G-dga A , $\mathsf{T}A^c$ is cofibrant in $\mathsf{dgCat}^{\text{cl}}$ (see the proof of Lem.2.2.4,(2)).*

The following gives a description of hom-sets of closed dg-categories in terms of dg-algebras and closed k -categories. This is used to translate a property of schematic homotopy types into a property of equivariant dg-algebras, see the proof of Prop.4.2.9.

Lemma 3.3.7. (1) *Let E be a closed dgc and $C \in \mathsf{dgCat}^{\text{cl}}/E$ be a closed dgc over E . Let $A = (G, A) \in \mathsf{dgAlg}_0^{\text{red}}$ be an object and $\mathsf{T}^{\text{ss}}A^c \longrightarrow E \in \mathsf{dgCat}^{\text{cl}}$ be a morphism. We consider $\mathsf{T}^{\text{ss}}A^c$ as an object of $\mathsf{dgCat}^{\text{cl}}/E$.*

For a morphism $\alpha : Z^0\mathsf{T}^{\text{ss}}A^c \rightarrow Z^0C \in \mathsf{Cat}^{\text{cl}}$ we define a dg-algebra $C_\alpha \in \mathsf{dgAlg}(G)$ as follows. We fix a presentation of $\mathcal{O}(G)_r$ as a union of finite dimensional G -representations: $\mathcal{O}(G)_r \cong \bigcup_\lambda V_\lambda$. We set

$$C_\alpha := \text{colim}_\lambda C(\alpha(V_\lambda)),$$

see the notation under Def.2.1.1. The algebra structure is defined from that of $\mathcal{O}(G)$ and the action of G is defined from the left translation on $\mathcal{O}(G)$. Then, there exists a bijection:

$$\text{Hom}_{\mathsf{dgCat}^{\text{cl}}/E}(\mathsf{T}^{\text{ss}}(A)^c, C) \cong \left\{ (\alpha, \beta) \mid \begin{array}{l} \alpha : Z^0\mathsf{T}^{\text{ss}}A^c \rightarrow Z^0C \in \mathsf{Cat}^{\text{cl}}/Z^0E, \\ \beta : A \rightarrow C_\alpha \in \mathsf{dgAlg}(G)/E_\alpha \end{array} \right\}$$

If the augmentation $a_C : C \rightarrow E \in \mathsf{dgCat}^{\text{cl}}$ is a fibration, this bijection induces a bijection:

$$[\mathsf{T}^{\text{ss}}A, C]_{\mathsf{dgCat}^{\text{cl}}/E} \cong \{(\alpha, \beta) | \alpha \in [Z^0\mathsf{T}^{\text{ss}}A, Z^0C]_{\mathsf{Cat}^{\text{cl}}/Z^0E}, \beta \in [A, C_\alpha]'_{\mathsf{dgAlg}(G)/E_\alpha}\}.$$

Here,

1. $[Z^0\mathsf{T}^{\text{ss}}A, Z^0C]_{\mathsf{Cat}^{\text{cl}}/Z^0E} := \text{Hom}_{\mathsf{Cat}^{\text{cl}}}(Z^0\mathsf{T}^{\text{ss}}A^c, Z^0C) / \sim$, where $\alpha_1 \sim \alpha_2$ if and only if there exists a natural isomorphism $t : \alpha_1 \Rightarrow \alpha_2$ such that t preserves tensors, i.e., $t(X \otimes Y) = t(X) \otimes t(Y)$, and $(a_C)_*t : a_C \circ \alpha_1 \Rightarrow a_C \circ \alpha_2 : Z^0\mathsf{T}^{\text{ss}}A^c \rightarrow Z^0E$ is the identity transformation.

2. $[A, C_\alpha]_{\mathrm{dgAlg}(G)/E_\alpha}' := [A, C_\alpha]_{\mathrm{dgAlg}(G)/E_\alpha} / \sim$, where $f_1 \sim f_2$ if and only if there exists a tensor preserving natural automorphism $t : \alpha \Rightarrow \alpha$ over E such that $t_{\alpha(\mathcal{O}(G))} \circ f_1 = f_2$.

(2) We use the notation of the previous part. Let $(G, A), (H, B) \in \mathrm{dgAlg}_{0,*}^{\mathrm{red}}$. Let $\alpha : Z^0 T^{\mathrm{ss}}(G, A)^c \rightarrow Z^0 T(H, B) \in \mathrm{Cat}_*^{\mathrm{cl}}$ be a morphism. Suppose α preserves semi-simple objects so that it induces a morphism $\alpha^* : H \rightarrow G$. Then, in the case $E = \mathrm{Vect}$,

$$[A, T(H, B)_\alpha]_{\mathrm{dgAlg}(G)/\mathrm{Vect}_\alpha} \cong [\alpha A, B]_{\mathrm{dgAlg}(H)/k}.$$

Here, αA denotes the pullback of A by $\alpha^* : H \rightarrow G$. In particular, if H is the one-element group,

$$[T(G, A), T(H, B)]_{\mathrm{dgCat}_*^{\mathrm{cl}}} \cong [A, B]_{\mathrm{dgAlg}/k}.$$

In the case $E = *$ (the terminal object of $\mathrm{dgCat}^{\mathrm{cl}}$),

$$[A, T(H, B)_\alpha]_{\mathrm{dgAlg}(G)}' \cong [\alpha A, B]_{\mathrm{dgAlg}(H)}/f \sim f * g \quad (=: [\alpha A, B]_{\mathrm{dgAlg}(H)}').$$

Here $f * g$ denotes the morphism $A \ni a \mapsto g \cdot a \mapsto f(g \cdot a) \in B$ and g runs through $C_{G(k)}(\alpha^* H(k))$, the centralizer of the image $\alpha^*(H(k))$ in $G(k)$.

Proof. (1) The bijection is defined by $(F : T^{\mathrm{ss}}(A)^c \rightarrow C) \longmapsto (Z^0 F, \mathrm{colim}_\lambda F_{(\mathbf{1}, V_\lambda)})$. Here $\mathrm{colim}_\lambda F_{(\mathbf{1}, V_\lambda)}$ denotes the following composition: $A \cong (\mathcal{O}(G)_r \otimes A)^G \cong \mathrm{colim}_\lambda (V_\lambda \otimes A)^G \cong \mathrm{colim} \mathrm{Hom}_{T^{\mathrm{ss}} A^c}(\mathbf{1}, V_\lambda) \xrightarrow{F_{V_\lambda}} \mathrm{colim}_\lambda \mathrm{Hom}_C(\mathbf{1}, \alpha(V_\lambda)) = C_\alpha$. The proof of the former claim is similar to Prop.3.3.4. The latter part follows from an explicit description of path objects in $\mathrm{dgCat}^{\mathrm{cl}}/E$ (see [30, 2.3.2]) and Lem.3.3.5, (1). In fact, right homotopies of closed dg-categories of the above forms can be decomposed into natural transformations of Z^0 and right homotopies of dg-algebras. Note that if $\alpha_1 \sim \alpha_2$, $E_{\alpha_1} = E_{\alpha_2}$ and $C_{\alpha_1} \cong C_{\alpha_2}$ in $\mathrm{dgAlg}(G)/E_{\alpha_1}$.
(2) We shall show morphisms $F : (\alpha^*)^* A \rightarrow T(H, B)_\alpha \in \mathrm{dgAlg}(G)$ are in one-to-one correspondence with morphisms $f : (\alpha^*)^* A \rightarrow B \in \mathrm{dgAlg}(H)$. For given F , we set $f = \mathcal{O}(\alpha^*)_* \circ F$, where $\mathcal{O}(\alpha^*) : \alpha(\mathcal{O}(G)) \rightarrow \mathcal{O}(H)$ is the induced H -module morphism and $\mathcal{O}(\alpha^*)_* : T(H, B)_\alpha \rightarrow B$ is corresponding push-forward. In the other direction, for given f , We set $F = Tf : A \cong TA(\mathcal{O}(G)) \rightarrow TB(\alpha(\mathcal{O}(G))) = TB_\alpha$. The construction $f \mapsto F \mapsto f$ is clearly the identity. We shall show $F \mapsto f \mapsto F$ is the identity. Let $a \in A$ and put $F(a) = \sum_i F_i(a) \otimes b_i(a)$, $F_i(a) \in \mathcal{O}(G)$, $b_i(a) \in B$. Put $\rho_A(a) = \sum_j a_j \otimes l_j$ and $\lambda_G(F_i(a)) = \sum_k F_{i,k}(a) \otimes m_{i,k}(a)$, where λ_G denotes the left translation, $a_j \in A$, $l_j, F_{i,k}(a)$, and $m_{i,k}(a) \in \mathcal{O}(G)$. As F is G -equivariant, $\sum_{i,j} F_i(a_j) \otimes b_i(a_j) \otimes l_j = \sum_{i,k} F_{i,k}(a) \otimes b_i(a) \otimes m_{i,k}(a)$. The constructed F is $\sum_{i,j} u \circ F_i(a_j) \cdot b_i(a_j) \otimes l_j$, where $u : \mathcal{O}(G) \rightarrow k$ is the counit map, and this is equal to $\sum_{i,k} u \circ F_{i,k}(a) \cdot b_i(a) \otimes m_{i,k}(a) = \sum_i F_i(a) \otimes b_i(a)$. This correspondence clearly preserves right homotopy relation so the former pointed claims follow from Lem.2.2.4. For the unpointed ($E = *$) claim, similarly to the above, we have a bijection: $[A, T(H, B)_\alpha]_{\mathrm{dgAlg}(G)} \cong [\alpha A, B]_{\mathrm{dgAlg}(H)}$. Under Tannakian duality, a (tensor-preserving) natural isomorphism $t : \alpha \Rightarrow \alpha$ corresponds an element g of $C_{G(k)}(\alpha^* H(k))$ and $t_{\alpha(\mathcal{O}(G))} \circ f$ does $f * g$ under the above bijection. \square

3.3.1 Iterated Hirsch extensions

We shall introduce the notion of iterated Hirsch extensions of dg-algebras. The corresponding notion in the trivial group case was considered by Sullivan, see [8, P.279-280]. In fact, an iterated Hirsch extension is an iteration of Hirsch extensions as we see below, but this notion is useful because its classifying data directly correspond to homotopy invariants of a space. Let $l \geq 1$. Let G be a reductive affine group scheme and $A \in \mathrm{dgAlg}(G)_0$ be a connected G -equivariant dg-algebra. Let $(W, \eta) \in TA$ be an object, where $W \in \mathrm{Ob}(\mathrm{Rep}(G))$ and η is a MC element on W (see subsection 2.2), and $\alpha \in Z^{l+1}(TA(W, \eta))$ be a cocycle of degree $l+1$ (see the notation under Def.2.1.1). We define a G -dga $A \otimes_{(\alpha, \eta)} \Lambda(W^\vee, l) \in \mathrm{dgAlg}_0^{\mathrm{red}}$ as follows

1. As an equivariant commutative gaded algebra,

$$A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l) = A \otimes_k \bigwedge(W^\vee, l),$$

where $\bigwedge(W^\vee, l)$ is the free commutative graded algebra generated by the dual space W^\vee with degree l and regarded as an G -equivariant algebra with the action induced by that on W .

2. The differential d is determined by its restrictions to A and W^\vee . $d|_A$ is equal to the differential of A . $d|_{W^\vee}$ is given by

$$\alpha \oplus (-^t\eta) : W^\vee \longrightarrow A^{l+1} \oplus A^1 \otimes_k W^\vee.$$

Explicitly, if we express α and η as

$$\eta = \sum_i f_i \otimes a_i \quad f_i \in \mathfrak{Hom}(W, W), \quad a_i \in A^1, \quad \alpha = \sum_j v_j \otimes b_j \quad v_j \in W, \quad b_j \in A^{l+1},$$

we set $[\alpha \oplus (-^t\eta)](u) = (\sum_j u(v_j) \cdot b_j, -\sum_i a_i \otimes (u \circ f_i))$. Maurer-Cartan condition on η and the condition $d_\eta \alpha = 0$ ensures $d^2 = 0$.

We always identify A with a subalgebra of $A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l)$ by $a \mapsto a \otimes 1$.

Definition 3.3.8. We use the above notations. Let $f : A \rightarrow B$ be a morphism of G -dga's. We say f is an iterated Hirsch extension of A if there exist data $(W, \eta) \in \text{Ob}(\mathsf{T}A)$, $\alpha \in Z^{l+1}(\mathsf{T}A(W, \eta))$, and an isomorphism $\varphi : B \rightarrow A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l)$ such that the following triangle is commutative.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow i & \downarrow \varphi \\ & & A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l), \end{array}$$

where i is the natural inclusion. We say a pair

$$\{ (W, \eta), [\alpha] \}$$

of an object $(W, \eta) \in \text{Ob}(\mathsf{T}A)$ and a class $[\alpha] \in H^{l+1}(\mathsf{T}A(W, \eta))$ is the classifying data of f (or B) if the above commutative triangle exists for (W, η) , α , and for some isomorphism φ . Classifying data is well-defined up to isomorphisms by Lem.3.3.9 below. If η can be chosen as 0, we say f is a Hirsch extension of A .

An iterated Hirsch extension $A \rightarrow A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l)$ is decomposable into a sequence of Hirsch extensions. Precisely, there exists a finite sequence of G -dga's

$$A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_m = A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l)$$

such that A_i is a Hirsch extension of A_{i-1} for $1 \leq i \leq m$. This is because the MC element η can be replaced with a MC-element of the upper triangular form with zero diagonal, up to isomorphism (see Lem.2.2.3). In particular, if A is minimal, $A \otimes_{(\alpha,\eta)} \bigwedge(W^\vee, l)$ is also minimal.

The following is a variant of [8, Thm.2.1] or [10, Lem.9.3].

Lemma 3.3.9. *Let G be a reductive affine group scheme.*

- (1) *Let A be a connected G -dga. Let $A \otimes_{(\alpha_i, \eta_i)} \bigwedge(W_i^\vee, l)$, $i = 0, 1$ be two iterated Hirsch extensions of A . There exists an isomorphism of G -dga's $\varphi : A \otimes_{(\alpha_0, \eta_0)} \bigwedge(W_0^\vee, l) \cong A \otimes_{(\alpha_1, \eta_1)} \bigwedge(W_1^\vee, l)$ fixing A if and only if there exists an isomorphism $\phi : (W_1, \eta_1) \cong (W_0, \eta_0) \in Z^0 \mathbf{T}A$ such that $[\phi \circ \alpha_1] = [\alpha_0] \in H^{l+1}(\mathbf{T}A(W_0, \eta_0))$.*
- (2) *Let \mathcal{M} be a minimal G -dga generated by elements of degree $\leq l$. If the l -th indecomposable module V of \mathcal{M} is finite dimensional, \mathcal{M} is an iterated Hirsch extension of the dg-subalgebra $\mathcal{M}(l-1)$ generated by elements of degree $\leq l-1$. Let $\{(W, \eta), [\alpha]\}$ be its classifying data. We identify V with a submodule of \mathcal{M}^l . As \mathcal{M} is minimal, its differential $d|_V$, restricted to V , has a unique decomposition $d|_V = d^{\mathcal{M}(l-1)} \oplus d^{\mathcal{M}^1 \otimes V}$ consisting of maps*

$$d^{\mathcal{M}(l-1)} : V \longrightarrow \mathcal{M}(l-1)^{l+1}, \quad d^{\mathcal{M}^1 \otimes V} : V \longrightarrow \mathcal{M}^1 \otimes V \in \text{Rep}^\infty(G).$$

Then,

1. *the dual V^\vee is isomorphic to W ,*
2. *$d^{\mathcal{M}^1 \otimes V}$ corresponds to the MC element η , and*
3. *$d^{\mathcal{M}(l-1)}$ corresponds to a cocycle α which gives the class $[\alpha]$.*

Proof. See [8, Thm.2.1]. □

3.4 de Rham functor

In this subsection, we define a Quillen adjoint pair

$$\mathbf{T}_{\text{PL}} : \text{sSet} \rightleftarrows (\text{dgCat}^{\text{cl}})^{\text{op}} : \langle - \rangle$$

and prove that $\mathbf{T}_{\text{PL}}(K)$ is a Tannakian dg-category if K is connected.

We first recall the notion of standard simplicial commutative dga $\nabla(*, *)$ over k from [6, Section 1]. Let $p \geq 0$ and $\nabla(p, *)$ be the commutative graded algebra over k generated by indeterminates t_0, \dots, t_p of degree 0 and dt_0, \dots, dt_p of degree 1 with relations

$$t_0 + \dots + t_p = 1, \quad dt_0 + \dots + dt_p = 0.$$

We regard $\nabla(p, *)$ as a dga with the differential given by $d(t_i) := dt_i$. We can define simplicial operators

$$d_i : \nabla(p, *) \rightarrow \nabla(p-1, *), \quad s_i : \nabla(p, *) \rightarrow \nabla(p+1, *), \quad 0 \leq i \leq p$$

(see [6]) and we also regard $\nabla(*, *)$ as a simplicial commutative dga.

The following definition is adopted in [16]

Definition 3.4.1 ([16]). *Let Vect^{iso} be the subcategory of Vect consisting of all objects and isomorphisms. Let K be a simplicial set.*

1. *A local system \mathcal{L} on K is a functor $(\Delta K)^{\text{op}} \rightarrow \text{Vect}^{\text{iso}}$.*
2. *A morphism of local systems $\mathcal{L} \rightarrow \mathcal{L}'$ is a natural transformation $I \circ \mathcal{L} \Rightarrow I \circ \mathcal{L}' : (\Delta K)^{\text{op}} \rightarrow \text{Vect}$, where $I : \text{Vect}^{\text{iso}} \rightarrow \text{Vect}$ is the natural inclusion functor.*

We define the tensor $\mathcal{L} \otimes \mathcal{L}'$, the internal hom object $\mathfrak{Hom}(\mathcal{L}, \mathcal{L}')$ and the coproduct $\mathcal{L} \oplus \mathcal{L}'$ of two local systems $\mathcal{L}, \mathcal{L}'$ by using those of Vect. For example, $(\mathcal{L} \otimes \mathcal{L}')(\sigma) = \mathcal{L}(\sigma) \otimes \mathcal{L}'(\sigma)$. We denote by $\text{Loc}(K)$ the closed k -category of local systems on K . If K is pointed, $\text{Loc}(K)$ is regarded as a closed k -category with a fiber functor. The fiber functor $\text{Loc}(K) \rightarrow \text{Vect}$ is given by the evaluation at the base point.

It is well-known that for a pointed connected simplicial set K , there exists an equivalence of closed k -categories $\text{Loc}(K) \xrightarrow{\sim} \text{Rep}(\pi_1(K))$ which is functorial in K . In the following, we sometimes identify k -local systems with representations of the fundamental group, fixing such an equivalence.

Definition 3.4.2. Let K be a simplicial set and \mathcal{L} be a local system on K . The de Rham complex of \mathcal{L} -valued polynomial forms $C_{\text{PL}}(K, \mathcal{L}) \in C^{\geq 0}(R)$ is defined as follows. For each $q \geq 0$, the degree q part is given by

$$C_{\text{PL}}^q(K, \mathcal{L}) = \lim_{\Delta K^{\text{op}}} \nabla(*, q) \otimes_k \mathcal{L}.$$

Here $\nabla(*, q)$ is regarded as a functor from ΔK^{op} to the category of k -vector spaces by composed with the functor $\Delta K^{\text{op}} \rightarrow \Delta^{\text{op}}$, the limit is taken in the category of possibly infinite dimensional k -vector spaces. For $q \leq -1$, we set $C_{\text{PL}}^q(K, \mathcal{L}) = 0$. The differential $d : C_{\text{PL}}^q(K, \mathcal{L}) \rightarrow C_{\text{PL}}^{q+1}(K, \mathcal{L})$ is defined from that of $\nabla(*, q)$.

We shall define the generalized de Rham functor

$$T_{\text{PL}} : \text{sSet} \longrightarrow (\text{dgCat}^{\text{cl}})^{\text{op}}.$$

This is a natural generalization of the de Rham functor of [6, Definition 2.1]. For $K \in \text{sSet}$ we define a closed dgc $T_{\text{PL}}(K)$ as follows. An object is a local system on K and $\text{Hom}_{T_{\text{PL}}(K)}(\mathcal{L}, \mathcal{L}') = C_{\text{PL}}(K, \mathfrak{Hom}(\mathcal{L}, \mathcal{L}'))$. The composition is defined from that of Vect and the multiplication of $\nabla(*, *)$, i.e.,

$$(\eta \cdot b) \circ (\omega \cdot a) := (\eta \cdot \omega) \cdot (b \circ a)$$

for $\omega, \eta \in \nabla(*, *)$, $a \in \mathfrak{Hom}(\mathcal{L}, \mathcal{L}')$ and $b \in \mathfrak{Hom}(\mathcal{L}', \mathcal{L}'')$. The additional structures \otimes , \mathfrak{Hom} and \oplus are defined similarly. (We agree that) $T_{\text{PL}}(\emptyset)$ is a terminal object of dgCat^{cl} . For each morphism $f : K \rightarrow L \in \text{sSet}$ we associate a morphism of closed dgc's $f^* : T_{\text{PL}}(L) \rightarrow T_{\text{PL}}(K)$ by

$$(\Delta(L)^{\text{op}} \xrightarrow{\mathcal{L}} \text{Vect}^{\text{iso}}) \longmapsto (\Delta(K)^{\text{op}} \xrightarrow{\Delta f^{\text{op}}} \Delta(L)^{\text{op}} \xrightarrow{\mathcal{L}} \text{Vect}^{\text{iso}}).$$

Thus we have defined a functor $T_{\text{PL}} : \text{sSet} \longrightarrow (\text{dgCat}^{\text{cl}})^{\text{op}}$.

Let $C \in \text{dgCat}^{\text{cl}}$. We define a functor $\langle - \rangle : (\text{dgCat}^{\text{cl}})^{\text{op}} \rightarrow \text{sSet}$ by

$$\langle C \rangle_n = \text{Hom}_{\text{dgCat}^{\text{cl}}}(C, T_{\text{PL}}(\Delta^n))$$

with obvious simplicial operators. Clearly $\langle - \rangle$ is a right adjoint of T_{PL} .

We define an adjoint pair between pointed categories:

$$T_{\text{PL}} : \text{sSet}_* \rightleftarrows (\text{dgCat}_*^{\text{cl}})^{\text{op}} : \langle - \rangle.$$

For $K \in \text{sSet}_*$, we define $T_{\text{PL}}(K) \in \text{dgCat}_*^{\text{cl}}$ by the following pullback square:

$$\begin{array}{ccc} T_{\text{PL}}(K) & \longrightarrow & T_{\text{PL}}(K_u) \\ \downarrow & & \downarrow T_{\text{PL}}(pt) \\ \text{Vect} & \longrightarrow & T_{\text{PL}}(*), \end{array}$$

where K_u is the unpointed simplicial set underlying K . For $(C, \omega_C) \in \text{dgCat}_*^{\text{cl}}$, we set $\langle C, \omega_C \rangle := \langle C \rangle$ whose base point is given by $* \cong \langle \text{Vect} \rangle \xrightarrow{\langle \omega_C \rangle} \langle C \rangle$.

Lemma 3.4.3 ([30]). *The above two adjoint pairs $(T_{PL}, \langle - \rangle) : s\text{Set} \rightarrow dg\text{Cat}^{\text{cl}}$ and $(T_{PL}, \langle - \rangle) : s\text{Set}_* \rightarrow dg\text{Cat}_*^{\text{cl}}$ are Quillen pairs.*

Definition 3.4.4 ($A_{\text{red}}(K)$). *Let K be a pointed connected simplicial set. We set $A_{\text{red}}(K) := A_{\text{red}}(T_{PL}(K)) \in dg\text{Alg}_{0,*}^{\text{red}}$. We always identify the affine group scheme of $A_{\text{red}}(K)$ with $\pi_1(K)^{\text{red}}$ (see Appendix A.2). When we want to clarify the field of definition, we write $A_{\text{red}}(K, k)$.*

We shall show the closed dg-category $T_{PL}(K)$ of a connected simplicial set K is a Tannakian dg-category. Recall that there is a natural equivalence $\mathsf{T}A^c \rightarrow \mathsf{T}A$ of closed dg-categories for each equivariant dga A (see sub-subsection 2.2.1).

Theorem 3.4.5. (1) *Let $K \in s\text{Set}$ (resp. $s\text{Set}_*$).*

1. $T_{PL}K$ is complete.

2. $0 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L}_1 \rightarrow 0$ is a short exact sequence in $\text{Loc}(K) \cong Z^0 T_{PL}(K)$ if and only if $\mathcal{L}_0 \rightarrow \mathcal{L}_2 \rightarrow \mathcal{L}_1$ is an extension in $T_{PL}K$ in the sense of subsection 2.2.

In particular, if K is connected, $T_{PL}K \in \mathsf{Tan}$ (resp. Tan_*).

(2) *Let K be a pointed connected simplicial set. There exists a morphism of $dg\text{Cat}_*^{\text{cl}}$ which is an equivalence between underlying categories:*

$$\mathsf{T}A_{\text{red}}(K)^c \longrightarrow T_{PL}K.$$

In particular, if $f : A \rightarrow A_{\text{red}}(K) \in dg\text{Alg}(\pi_1^{\text{red}})$ is a quasi-isomorphism ($\pi_1 = \pi_1(K)$), finite dimensional representations V of π_1 determine and are determined by objects $(W, \eta) \in \text{Ob}(\mathsf{T}A)$ up to isomorphisms via the following equivalences:

$$\text{Rep}(\pi_1) \simeq \text{Loc}(K) \simeq Z^0 T_{PL}(K) \simeq Z^0 \mathsf{T}A_{\text{red}}(K)^c \simeq Z^0 \mathsf{T}A_{\text{red}}(K) \xrightarrow{f_*} Z^0 \mathsf{T}A.$$

In this correspondence, W is isomorphic to the semi-simplification of V .

Proof. First note that the following facts.

1. The completeness is stable under homotopy pullbacks and homotopy limits of towers.
2. For a dgc C , a sequence of chain morphisms $c_0 \xrightarrow{a} c_2 \xrightarrow{b} c_1$ in C is an extension if and only if for any object $c \in \text{Ob}(C)$ the two sequences of complexes

$$\begin{aligned} 0 \rightarrow \text{Hom}_C(c, c_0) &\xrightarrow{a_*} \text{Hom}_C(c, c_2) \xrightarrow{b_*} \text{Hom}_C(c, c_1) \rightarrow 0, \\ 0 \rightarrow \text{Hom}_C(c_1, c) &\xrightarrow{b^*} \text{Hom}_C(c_2, c) \xrightarrow{a^*} \text{Hom}_C(c_0, c) \rightarrow 0 \end{aligned}$$

are both levelwise exact.

By these facts and small object argument, all we have to do is to prove the proposition for $X = \Delta^n, \partial\Delta^n$ for $n \geq 1$. We only show the case of $T_{PL}(\partial\Delta^2)$ and the others are clear. We may replace $\partial\Delta^2$ by $S^1 = \Delta^1/0 \sim 1$ and we identify local systems on S^1 with representation of the free group \mathbb{Z} . Let $(V, g), (V', g') \in \text{Rep}(\mathbb{Z})$. Let $\Sigma_i P_i(t)dt \cdot f_i \in \text{Hom}_{T_{PL}S^1}^1((V, g), (V', g'))$ where $t = t_0$, $P_i(t)$ is a polynomial of t , and $f_i \in \text{Hom}(V, V')$. Put

$$g_0 = \begin{pmatrix} g' & \Sigma_i \int_0^1 P_i(t)dt \cdot f_i g \\ 0 & g \end{pmatrix}.$$

Then the obvious sequence $(V', g') \rightarrow (V' \oplus V, g_0) \rightarrow (V, g)$ is an extension whose class is equal to $[\Sigma_i P_i(t)dt \cdot f_i]$.

So $T_{PL}(S^1)$ is complete and the third condition of Def.3.1.1 is proved by a similar argument.

(2) follows from (1) and Prop.3.3.4. \square

The following lemma is used in the proof of Thm.4.3.4.

Lemma 3.4.6 (cf. Rem.4.43 of [27]). *Let G be a reductive affine group scheme and \mathcal{M} be a minimal G -dga. Let V^i denote the i -th indecomposable module of \mathcal{M} . For $i \geq 2$, there exists an isomorphism of groups $\pi_i(\mathbb{R}\langle T\mathcal{M} \rangle) \cong (V^i)^\vee$. This isomorphism is functorial about morphisms between minimal equivariant dga.*

Proof. By Lem.2.2.4 and 3.3.7, $\pi_i(\mathbb{R}\langle T\mathcal{M} \rangle) \cong [T\mathcal{M}, TPLS^i]_{dg\mathbf{Cat}_*^{cl}} \cong [\mathcal{M}, A_{PL}S^i]_{dg\mathbf{Alg}/k} \cong (V^i)^\vee$. \square

4 The de Rham homotopy theory for general spaces

In this section, we see how the minimal models describe algebraic topological invariants of spaces and provide some examples of minimal models. We also prove an equivalence between Tannakian dg-categories with subsidiary data and fiberwise rationalizations. In the proofs of results of this section, we use the correspondence between Tannakian dg-categories and schematic homotopy types (Thm.5.2.1 and Cor.5.2.7). As for logical order, section 5 is previous to this section.

4.1 Homotopy invariants

We shall recall the notion of algebraically goodness introduced by Toën [22].

Let Γ be a discrete group. Let $H^i(\Gamma, -)$ be the i -th derived functor of invariants

$$\text{Rep}_k^\infty(\Gamma) \longrightarrow k\text{-Mod}, \quad V \longmapsto V^\Gamma,$$

and $H^i(\Gamma^{\text{alg}}, -)$ be the i -th derived functor of the functor

$$\text{Rep}_k^\infty(\Gamma^{\text{alg}}) \longrightarrow k\text{-Mod}, \quad V \longmapsto V^{\Gamma^{\text{alg}}}.$$

Any Γ^{alg} -module can be regarded as a Γ -module by pulling back by the canonical map $\Gamma \rightarrow \Gamma^{\text{alg}}(k)$ so there exists a canonical natural transformation

$$H^i(\Gamma^{\text{alg}}, -) \Longrightarrow H^i(\Gamma, -) : \text{Rep}_k^\infty(\Gamma^{\text{alg}}) \longrightarrow k\text{-Mod}.$$

Definition 4.1.1 (algebraically good, [22, 25, 27]). *Under above notations, we say Γ is algebraically good over k if for each $i \geq 0$ and each finite dimensional representation $V \in \text{Rep}_k(\Gamma^{\text{alg}})$, the canonical map*

$$H^i(\Gamma^{\text{alg}}, V) \longrightarrow H^i(\Gamma, V)$$

is an isomorphism.

The following was proved by Pridham [27].

Theorem 4.1.2 ([27]). *Let Γ be a discrete group. Γ is algebraically good over k if and only if the minimal model of $A_{\text{red}}(K(\Gamma, 1), k)$ is generated by elements of degree 1.*

Proof. Clearly, Γ is algebraically good if and only if the canonical map $A_{\text{red}}(K(\Gamma^{\text{alg}}, 1)) \rightarrow A_{\text{red}}(K(\Gamma, 1))$ is a quasi-isomorphism (see subsection 5.2). But by [25, Prop.4.12], for any discrete group Γ the map $H^i(A_{\text{red}}(K(\Gamma^{\text{alg}}, 1))) \rightarrow H^i(A_{\text{red}}(K(\Gamma, 1)))$ is an isomorphism for $i = 0, 1$ and a monomorphism for $i = 2$ and by Cor.5.2.6, the minimal model of $A_{\text{red}}(K(\Gamma^{\text{alg}}, 1))$ is generated by degree one elements so the claim follows. \square

We shall show how the minimal model describes homotopy theory of a space.

Theorem 4.1.3. Let K be a pointed connected simplicial set. Put $\pi_i := \pi_i(K)$. For $i \geq 2$, we regard π_i as a π_1 -module by the canonical action. Let

1. $K \rightarrow \cdots \xrightarrow{p_i} K_{(i-1)} \xrightarrow{p_{i-1}} \cdots \xrightarrow{p_1} K_{(1)}$ be the Postnikov tower of K ,
2. \mathcal{M} be the minimal model of $A_{\text{red}}(K)$,
3. $\mathcal{M}(i)$ be the dg-subalgebra of \mathcal{M} generated by $\oplus_{j \leq i} \mathcal{M}^j$, and
4. $n \geq 2$ be an integer.

Suppose π_1 is algebraically good over k and π_i is of finite rank as an Abelian group for each $2 \leq i \leq n$.

(1) There exists a commutative diagram in Tan_*

$$\begin{array}{ccccccc} \mathsf{T}\mathcal{M}(1)^c & \xrightarrow{\mathsf{T}l_1^c} & \mathsf{T}\mathcal{M}(2)^c & \xrightarrow{\mathsf{T}l_2^c} & \cdots & \xrightarrow{\mathsf{T}l_{n-1}^c} & \mathsf{T}\mathcal{M}(n)^c \\ q_1 \downarrow & & q_2 \downarrow & & & & q_n \downarrow \\ \mathsf{T}_{\text{PL}}K_{(1)} & \xrightarrow{p_1^*} & \mathsf{T}_{\text{PL}}K_{(2)} & \xrightarrow{p_2^*} & \cdots & \xrightarrow{p_{n-1}^*} & \mathsf{T}_{\text{PL}}K_{(n)} \end{array}$$

such that $l_i : \mathcal{M}(i) \rightarrow \mathcal{M}(i+1)$ is the inclusion and all the vertical arrows are quasi-equivalences.

(2) For each $2 \leq i \leq n$, the inclusion $l_{i-1} : \mathcal{M}(i-1) \rightarrow \mathcal{M}(i)$ is an iterated Hirsch extension (see Def.3.3.8). Let $\{(W_i, \eta_i), [\alpha_i]\}$ be its classifying data. Then,

1. The object $(W_i, \eta_i) \in \text{Ob}(\mathsf{T}\mathcal{M}(i-1)) = \text{Ob}(\mathsf{T}\mathcal{M})$ is isomorphic to the π_1 -representation $\pi_i \otimes_{\mathbb{Z}} k$ under the correspondence of Thm.3.4.5,(2).
2. The class $[\alpha_i] \in H^{i+1}(\mathsf{T}\mathcal{M}(i-1)(W_i, \eta_i))$ corresponds to the k -invariant tensored with k via the isomorphism:

$$H^{i+1}[\mathsf{T}\mathcal{M}(i-1)(W_i, \eta_i)] \cong H^{i+1}[\mathsf{T}_{\text{PL}}(K_{(i-1)})(\pi_i \otimes_{\mathbb{Z}} k)] \cong H^{i+1}(K_{(i-1)}; \pi_i \otimes_{\mathbb{Z}} k).$$

induced by q_{i-1} in the above diagram. Here, $\pi_i \otimes_{\mathbb{Z}} k$ is considered as a local system on $K_{(i-1)}$.

See Lem.3.3.9,(2). See also [27, Thm.1.58, Rem.4.43].

Before we begin the proof, we state a corollary which may be useful for computations.

Corollary 4.1.4. Under the assumption of Thm.4.1.3, suppose the action of π_1 on $\pi_n \otimes_{\mathbb{Z}} k$ is semisimple. Then, $\mathcal{M}(n)$ is a Hirsch extension of $\mathcal{M}(n-1)$. (Note that we do not assume $\mathcal{M}^1 = 0$.)

To prove the theorem, we need to show a variant of the Hirsch Lemma [10, Thm.11.1] formulated in the following.

Let $m \geq 2$. We shall consider a fibration

$$p : E \longrightarrow B$$

between pointed connected simplicial sets whose fiber F satisfies $\pi_0(F) \cong *$, $\pi_i(F) = 0$ for $i \leq m-1$, and $\pi_m(F)$ is an Abelian group of finite rank. Put $\pi_1 := \pi_1(B) \cong \pi_1(E)$. Let π denote the local system of the n -th k -tensored homotopy groups of fibers of p or the corresponding representation of $\pi_1(B)$. Then

$$H^{m+1}(B, E; \pi) \cong \text{Hom}_{\text{Rep}(\pi_1(B))}(\pi_{m+1}(B, E), \pi) \cong \text{Hom}_{\text{Rep}(\pi_1(B))}(\pi, \pi).$$

See [15, P.344] or [9, P.289]. Take the element \tilde{k} of $H^{m+1}(B, E; \pi)$ corresponding to the identity on π via this isomorphism. By definition, the image of \tilde{k} in $H^{m+1}(B; \pi)$ is the k -invariant of p tensored with k . Let

\mathcal{M}_B be a minimal model of $A_{\text{red}}(B)$. We may replace $TPL(B)$ by $T\mathcal{M}_B$ and we have $H^{m+1}(B, E; \underline{\pi}) \cong H^{m+1}(\text{Cone}(p^* : CPL(B; \underline{\pi}) \rightarrow CPL(E; \underline{\pi})) \cong H^{m+1}(\text{Cone}(p^* : T\mathcal{M}_B(\pi^{\text{ss}}, \eta) \rightarrow CPL(E; \pi)))$. Here, (π^{ss}, η) is a pair of the semisimplification of π and a MC-element η , see Thm.3.4.5. We take a cocycle $(\alpha, \beta) \in \text{Cone}^{m+1}(p^* : T\mathcal{M}_B(\pi^{\text{ss}}, \eta) \rightarrow CPL(E; \pi))$, where $\alpha \in T\mathcal{M}_B(\pi^{\text{ss}}, \eta)^{m+1}$ and $\beta \in C_{\text{PL}}^m(E; \underline{\pi})$ which represents \tilde{k} .

Take the iterated Hirsch extension $\mathcal{M}_B \otimes_{(\alpha, \eta)} \bigwedge((\pi^{\text{ss}})^{\vee}, m)$. p and β define a morphism of π_1^{red} -dga's:

$$\varrho : \mathcal{M}_B \otimes_{(\alpha, \eta)} \bigwedge((\pi^{\text{ss}})^{\vee}, m) \longrightarrow A_{\text{red}}(E)$$

by $\varrho|_{\mathcal{M}_B} = p^*$ and $\rho|_{(\pi^{\text{ss}})^{\vee}} = \beta$ (see Def.3.3.8). Here β is considered as a π_1^{red} -module homomorphism $(\pi^{\text{ss}})^{\vee} \longrightarrow A_{\text{red}} E$.

The essence of the proof of the following is the same as that of [10, Thm.11.1].

Lemma 4.1.5. *We use the above notations.*

- (1) *ϱ induces an isomorphism between i -th cohomology groups for each $i \leq m$ and a monomorphism between $m+1$ -th cohomology groups.*
- (2) *If a fiber F of p satisfy $\pi_i(F) = 0$ for $i \geq m+1$, the map ϱ is a quasi-isomorphism.*

Proof. We prove (2). The proof of (1) is similar and easier. In the following, we use cubical sets instead of simplicial sets. For details about cubical de Rham theory, see Appendix A.1. We use the same notation as the case of simplicial sets for the corresponding notion in the cubical case. Put $A_B := \mathcal{M}_B \otimes_{(\alpha, \eta)} \bigwedge((\pi^{\text{ss}})^{\vee}, m)$. We define a descending filtration

$$A_B = F_B^0 \supset F_B^1 \supset \cdots \supset F_B^p \supset \cdots$$

by $F_B^p = \bigoplus_{i \geq p} \mathcal{M}_B^p \otimes_k \bigwedge((\pi^{\text{ss}})^{\vee}, n)$ i.e., F_B^p is the ideal of A_B generated by $\bigoplus_{i \geq p} \mathcal{M}_B^p$. This filtration induces a filtration of closed dg-category consisting of ideals closed under tensor and internal hom: $TA_B = TF_B^0 \supset TF_B^1 \supset \cdots \supset TF_B^p \supset \cdots$. Note that $TF_B^p \circ TF_B^q$, $TF_B^p \otimes TF_B^q$, $\mathfrak{Hom}(TF_B^p, TF_B^q) \subset TF^{p+q}$. On the other hand, we define a filtration $\{F^p(TPL E)\}_{p \geq 0}$ of $TPL(E)$ by $F^p(TPL E)$ being the $(\otimes, \mathfrak{Hom})$ -closed ideal generated by images of homogeneous morphism in $T\mathcal{M}_B$ of degree $\geq p$, by $p^* : TPL B \longrightarrow TPL E$. We may describe $F^p(TPL E)$ as follows. Let $\sigma \in E$ be a non-degenerate l -cube. We identify a cube with the sub-cubical set generated by it. For notational simplicity, we assume $p|_{\sigma} : \sigma \rightarrow p(\sigma)$ is the projection to the first k components $q_k : \square^l \rightarrow \square^k$. Indeed, after change of coordinate, $p|_{\sigma}$ is isomorphic to q_k for some k . Then

$$\omega \in \text{Hom}_{F^p(TPL E)}(\mathcal{L}, \mathcal{L}') \iff \forall \sigma \quad \omega|_{\sigma} \in \square(k, \geq p) \otimes_k \square(l-k, *) \otimes_k \mathfrak{Hom}(\mathcal{L}, \mathcal{L}')(\sigma).$$

Here, we identify $\square(l, *)$ with $\square(k, *) \otimes_k \square(l-k, *)$. Note that $F^p(TPL E) \circ F^{p'}(TPL E)$, $F^p(TPL E) \otimes F^{p'}(TPL E)$, $\mathfrak{Hom}(F^p(TPL E), F^{p'}(TPL E)) \subset F^{p+p'}(TPL E)$. Let $T_{\square}(E) \in \text{dgCat}^{\geq 0}$ be a dgc defined as follows.

1. $\text{Ob}(T_{\square}(E)) = \text{Loc}(E)$.
2. $\text{Hom}_{T_{\square}(E)}(\mathcal{L}, \mathcal{L}') = C_{\square}(E; \mathfrak{Hom}(\mathcal{L}, \mathcal{L}'))$ (see Appendix A.1) and the composition is given by the cup-product.

We define a filtration $\{F^p(T_{\square}(E))\}$ on $T_{\square}(E)$ as usual, by

$$\text{Hom}_{F^p(T_{\square}(E))}(\mathcal{L}, \mathcal{L}') = \text{Ker}(i^* : \text{Hom}_{T_{\square} E}(\mathcal{L}, \mathcal{L}') \rightarrow \text{Hom}_{T_{\square} E^{(p-1)}}(\mathcal{L}, \mathcal{L}')),$$

where $E^{(p-1)} = p^{-1}(B^{p-1})$ and $i : E^{(p-1)} \rightarrow E$ is the inclusion (B^{p-1} is the $p-1$ -th skeleton of B). ϱ induces filtration-preserving morphism

$$T\varrho : TA_B \longrightarrow TA_{\text{red}} E \simeq TPL E.$$

Stokes map

$$\rho : \mathrm{T}_{\mathrm{PL}} E \rightarrow \mathrm{T}_{\square} E$$

(this is a morphism of dg-graphs) also preserves filtration.

In the following, for a (closed) dgc C with a filtration $\{F^p(C)\}$, $E_r(C)$ denotes a (closed) dgc defined by

1. $\mathrm{Ob}E_r(C) = \mathrm{Ob}C$.
2. $\mathrm{Hom}_{E_r(C)}(-, -) = E_r(\mathrm{Hom}_C(-, -); \mathrm{Hom}_{F^\bullet(C)}(-, -))$, the E_r -term of the spectral sequence, with the differential d_r .

To prove the lemma, it is enough to prove $E_2(\rho \circ \mathrm{T}\varrho)$ induces isomorphisms between each hom-complexes. Note that there is a diagram

$$\begin{array}{ccc} E_1 \mathrm{T}_{\mathrm{PL}}(E) & \xrightarrow{\varphi} & \mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F) \\ & \searrow E_1 \rho & \downarrow \rho' \\ & & E_1 \mathrm{T}_{\square} E \end{array}$$

Here,

1. $\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F)$ is a closed dgc defined by
 - (a) $\mathrm{Ob}\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F) = \mathrm{Ob}(\mathrm{Loc}(B))$,
 - (b) $\mathrm{Hom}_{\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F)}(-, -) = \bigoplus_{p+q=n} \mathrm{C}_{\mathrm{PL}}^p(B; \mathcal{H}_F^q \otimes \mathfrak{Hom}(-, -))$, where \mathcal{H}_F^q is a local system on B given by $\tau \mapsto H^q(p^{-1}(\tau(0, \dots, 0)), \mathbb{Q})$,
2. ρ' is the Stokes map with $\mathcal{H}_F^q \otimes \mathfrak{Hom}(-, -)$ -coefficients (Here, we used the well-known identification $\mathrm{Hom}_{E_1 \mathrm{T}_{\square} E}^{p,q}(-, -) \cong \mathrm{C}_{\square}^p(B; \mathcal{H}_F^q \otimes \mathfrak{Hom}(-, -))$, and
3. φ is a morphism of closed dgc's defined as follows. An element $x \in \mathrm{Hom}_{E_0(\mathrm{T}_{\mathrm{PL}} E)}^{p,q}(-, -)$ defines a form ω on E such that $\omega|_\sigma \in \square(k, p) \otimes_k \square(n-k, q)$ (with the above notations). Let $\tau \in B$ be a non-degenerate k -cube. We take a lift $\tilde{\tau}$ of the following diagram

$$\begin{array}{ccc} F_\tau & \longrightarrow & E \\ \downarrow & \nearrow \tilde{\tau} & \downarrow \\ \square^k \times F_\tau & \xrightarrow{\tau} & B \end{array}$$

Here $F_\tau = p^{-1}(\tau(0, \dots, 0))$. Fixing basis of $\square(k, p)$, $\alpha_1, \dots, \alpha_N$, we can write $\tilde{\tau}^*(\omega) = \alpha_1 \otimes \beta_1 + \dots + \alpha_N \otimes \beta_N$, $\beta_i \in \mathrm{C}_{\mathrm{PL}}^q(F_\tau; \mathfrak{Hom}(-, -)|_\tau)$. So if x is a cocycle, it gives an element $\varphi(x) \in \mathrm{C}_{\mathrm{PL}}^p(B; \mathcal{H}_F^q \otimes \mathfrak{Hom}(-, -))$ defined as $\alpha_1 \otimes [\beta_1] + \dots + \alpha_N \otimes [\beta_N]$ on τ .

The Fubini's theorem ensures the diagram is commutative. It is easy to see $H^* \rho' : H^* \mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F) \rightarrow E_2 \mathrm{T}_{\square} E$ induces bijections of hom-sets, so all we have to do is to prove $H^*(\varphi \circ E_1 \mathrm{T}\varrho) = H^*(\varphi) \circ E_2(\mathrm{T}\varrho) : E_2(\mathrm{T}A_B) \rightarrow H^* \mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F)$ is an equivalence. We prove this by using the fact that the map $H^*(\varphi) \circ E_2(\mathrm{T}\varrho)$ is a morphism of closed dgc's. (This is analogous to the fact that in simply connected case, the corresponding claim was proved by using the fact that the corresponding map is a morphism of algebras.)

We need the following sub-lemma. We use the notation that $E_r^{p,q}(C)(X) := \mathrm{Hom}_{E_r(C)}^{p,q}(\mathbf{1}, X)$.

Sub-lemma 4.1.6. *Let $U^q \in \mathrm{Ob} \mathrm{T}A_B$ be \mathcal{H}_F^q regarded as an object of $\mathrm{T}A_B$.*

1. There exists a bijection $E_2^{p,q}(\mathrm{TA}_B)(V) \cong \mathrm{H}^p[\mathrm{TM}_B((\wedge \underline{\pi}^\vee)^q \otimes V)]$ for each $V \in \mathrm{Ob}(\mathrm{TA}_B)$.
2. $\mathrm{H}^*(\varphi) \circ E_2(\mathrm{T}\varrho)$ induces isomorphisms on $(p, 0)$ -terms of each hom-complex for $p \geq 0$.
3. $e \in E_2(\mathrm{TA}_B)^{0,m}((U^m)^\vee) \cong \mathrm{Hom}_{\mathrm{Rep}(\pi_1)}(U^m, \underline{\pi}^\vee)$ be an element corresponding to an isomorphism, and $\mathrm{ev}_{U^q} \in \mathrm{Hom}_{E_2(\mathrm{TA}_B)}^{0,0}(U^{q\vee} \otimes U^q, \mathbf{1})$ be the evaluation. Then the maps

$$\begin{aligned} E_2^{p,0} \mathrm{TA}_B(U^{km} \otimes V) &\longrightarrow E_2^{p,km} \mathrm{TA}_B(V) & x &\longmapsto \mathrm{ev}_{U^{km}} \circ (e^k \otimes x) \\ \mathrm{H}^p[\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F^0)(\psi(U^{km}) \otimes \mathcal{L})] &\longrightarrow \mathrm{H}^p[\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F^{km})(\mathcal{L})] & y &\longmapsto \mathrm{ev}_{\psi(U^{km})} \circ (\psi(e^k) \otimes y) \end{aligned}$$

are bijections, where $k \geq 1$, $e^k \in E_2(\mathrm{TA}_B)^{0,km}((U^k m)^\vee)$ is the k -times e , and $\psi = H^*(\varphi) \circ E_2(\mathrm{T}\varrho)$.

Proof of Sub-lemma. $E_0^{p,q}(\mathrm{TA}_B)(V)$ and $E_1^{p,q}(\mathrm{TA}_B)(V)$ are naturally isomorphic to $[(\wedge \underline{\pi}^\vee)^q \otimes V]^{\mathrm{ss}} \otimes \mathcal{M}_B^p]^{\pi_1^{\mathrm{red}}}$ and d_1 is equal to $d_{\mathcal{M}_B} + \eta_{(\wedge \underline{\pi}^\vee)^q \otimes V}$ ($\eta_{(\wedge \underline{\pi}^\vee)^q \otimes V}$ is the MC-element of $(\wedge \underline{\pi}^\vee)^q \otimes V$) so the first part follows. The second part is clear. For the third part, the first map is identified with the pushforward $e_*^k : \mathrm{H}^p[\mathrm{TM}_B((\wedge \underline{\pi}^\vee)^{km} \otimes V)] \rightarrow \mathrm{H}^p[\mathrm{TM}_B(U^{km} \otimes V)]$ via the bijection of part 1, so this is a bijection. To see the second map, note that $\mathrm{T}\varrho(e) = {}^t e \circ \beta \in \mathrm{C}_{\mathrm{PL}}^m(E; (\mathcal{H}_F^m)^\vee)$. The restriction of this element to a fiber is a cocycle in $\mathrm{C}_{\mathrm{PL}}^m(F; (\mathcal{H}_F^m|_F)^\vee)$ and it represents an isomorphism in $\mathrm{Hom}_{\mathrm{Rep}(\pi_1)}(\mathcal{H}_F^m|_F, \mathrm{H}^m(F)) \subset \mathrm{H}^m(F; (\mathcal{H}_F^m|_F)^\vee)$. Thus $\psi(e)$ corresponds to an isomorphism via the identification $\mathrm{H}^0[\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F^m)((\mathcal{H}_F^m)^\vee)] \cong \mathrm{H}^0(B; \mathcal{H}_F^m \otimes (\mathcal{H}_F^m)^\vee) \cong \mathrm{Hom}_{\mathrm{Loc}(B)}(\mathcal{H}_F^m, \mathcal{H}_F^m)$ so we can see the second map is a bijection similarly to the first one. \square

As $H^*(\varphi) \circ E_2(\mathrm{T}\varrho)$ is a morphism of closed dgc's, the following diagram is commutative.

$$\begin{array}{ccc} E_2^{p,0} \mathrm{TA}_B(U^{km} \otimes V) & \longrightarrow & E_2^{p,km} \mathrm{TA}_B(V) \\ \downarrow \psi & & \downarrow \psi \\ \mathrm{H}^p[\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F^0)(\psi(U^{mk}) \otimes \psi V)] & \longrightarrow & \mathrm{H}^p[\mathrm{T}_{\mathrm{PL}}(B; \mathcal{H}_F^{mk})(\psi V)], \end{array}$$

where the horizontal maps are the ones in the sub-lemma. This implies that ψ is an equivalence of categories. \square

Proof of Thm.4.1.3. The proof is successive applications of Thm.4.1.2 and Lem.4.1.5. Lem.4.1.5,(1) is necessary to prove $\mathcal{M}(n)$ is isomorphic to the minimal model of $\mathrm{A}_{\mathrm{red}}(K_{(n)})$, see [6, Prop.7.10]. \square

For the first infinite higher homotopy group, we have the following.

Theorem 4.1.7. *Let K be a pointed connected simplicial set with $\pi_1(K)$ algebraically good. We use the notation of Thm.4.1.3. Let $n \geq 2$ and suppose $\pi_i(K)$ is of finite rank as an abelian group for $2 \leq i \leq n-1$. As the action of $\pi_1(K)^{\mathrm{red}}$ on V^n is locally finite and as \mathcal{M} is minimal, there exist finite dimensional $\pi_1(K)^{\mathrm{red}}$ -submodules $\{V_\lambda^n\}_\lambda$ of V^n such that $\bigcup_\lambda V_\lambda = V^n$, $d^{\mathcal{M}^1 \otimes V^n}(V_\lambda^n) \subset M^1 \otimes V_\lambda^n$, and $(\{V_\lambda^n\}_\lambda, \subset)$ forms a filtered system. The restriction of $d^{\mathcal{M}^1 \otimes V^n}$ to V_λ^n defines a MC-element η_λ on $(V_\lambda^n)^\vee$ in $\mathrm{T}^{\mathrm{ss}}\mathcal{M}$ (see Lem.3.3.9). Thus we obtain an inverse system of finite dimensional $\pi_1(K)$ -representations corresponding to $\{(V_\lambda^n)^\vee, \eta_\lambda\}$.*

Then, the limit of the inverse system is isomorphic to the pro-finite dimensional completion of $\pi_n(K) \otimes_{\mathbb{Z}} k$. Here, the pro-finite dimensional completion of (possibly infinite dimensional) $\pi_1(K)$ -representation X is the limit of the inverse system $\{W_\mu \in \mathrm{Rep}(\pi_1(K)) | \mu : X \rightarrow W_\mu \in \mathrm{Rep}^\infty(\pi_1(K))\}$ taken in $\mathrm{Rep}^\infty(\pi_1(K))$.

Proof. The proof is similar to that of Thm.4.1.3 so we omit. \square

4.2 Examples

Definition 4.2.1. We denote by $\mathcal{M}_{\text{red}}(K)$ (resp. $\mathcal{M}_{\text{PL}}(K)$) the minimal model of $A_{\text{red}}(K)$ (resp. $A_{\text{PL}}(K)$). Here $A_{\text{PL}}(K)$ is the usual polynomial de Rham algebra over k . When we want to clarify the field of definition, we write $A_{\text{red}}(K, k)$, $\mathcal{M}_{\text{red}}(K, k)$ and $\mathcal{M}_{\text{PL}}(K, k)$.

Example 4.2.2. Let X be a pointed connected simplicial set with $\pi_1(X)$ finite. Then $\mathcal{M}_{\text{red}}(X) \cong \mathcal{M}_{\text{PL}}(\tilde{X})$, where \tilde{X} denotes the universal covering of X and the action of π_1 on $\mathcal{M}_{\text{PL}}(\tilde{X})$ is induced from the one on \tilde{X} (see [30]).

The method of proof of Lem.4.1.5 provide some examples.

Theorem 4.2.3. Let $K \in \text{sSet}_*$ be a nilpotent simplicial set of finite type. Then $\mathcal{M}_{\text{red}}(K) \cong \mathcal{M}_{\text{PL}}(K)$. Here $\mathcal{M}_{\text{PL}}(K)$ is considered as $\pi_1(K)^{\text{red}}$ -dga with the trivial action.

Proof. It is enough to show $\mathcal{M}_{\text{red}}(K(N, 1)) \cong \mathcal{M}_{\text{PL}}(K(N, 1))$ for a nilpotent group N . The proof is similar to that of Lem.4.1.5 and we use the tower of fibrations associated to nilpotent extensions instead of the Postnikov tower. A slight difference is that $\pi_1(B) \neq \pi_1(E)$ in this case. But in fact, for a finite dimensional $\pi_1(E)$ -module V , the E_2 -terms concerning V is naturally isomorphic to those concerning the $\pi_1(B)$ -module $V^{\pi_1(F)}$ of $\pi_1(F)$ -invariants, where F denotes the fiber, so the proof of Lem.4.1.5 still works. \square

Remark 4.2.4. If we express Thm.4.2.3 in the language of schematic homotopy types, there exists an isomorphism of schematic homotopy types:

$$(L \otimes k)^{\text{sch}} \cong (L \otimes k)^{\text{uni}} \times K(\pi_1(L)^{\text{red}}, 1) \in \text{Ho}(\text{SHT})$$

for nilpotent L of finite type (see [22] for notation).

Example 4.2.5. Suppose k is algebraically closed. Let $n \geq 2$ be an integer and N be the free abelian group of rank l and Let M be a finite rank abelian group with N -action. Let $K = K(N, M, n) := K(N \times K(M, n-1), 1)$. Here, $K(M, n-1)$ is an Eilenberg Maclane space realized as simplicial abelian group with the induced action of N . Let $g_j \in GL(M \otimes_{\mathbb{Z}} k)$ be the action of j -th generator of N and $g_j = g_j^s + g_j^n$ be a Jordan decomposition commutative with each other, where g_j^n is nilpotent and g_j^s is semisimple. Note that $\sum_{j=1, \dots, l} g_j^n \cdot s_j$ is a MC-element corresponding to $M \otimes_{\mathbb{Z}} k$ and the k -invariant is zero as the section $K(N, 1) \rightarrow K(N, M, n)$ exists. When we denote the module of i -dimensional generators of $\mathcal{M}_{\text{red}}(K, k)$ by V^i , by Thm.4.1.3,

$$V^i = \begin{cases} \bigoplus_{j=1, \dots, l} k \cdot s_j & (i = 1) \\ ((M \otimes_{\mathbb{Z}} k)^{\text{ss}})^{\vee} & (i = n) \\ 0 & (\text{otherwise}) \end{cases}$$

and $d(s_j) = 0$, $d(x) = \sum_{j=1, \dots, l} {}^t g_j^n(x) \cdot s_j$ for $x \in V^n$. Here, $(M \otimes_{\mathbb{Z}} k)^{\text{ss}}$ is the semisimplification of N -representation $M \otimes_{\mathbb{Z}} k$, i.e., the j -th generator acts on it by g_j^s . N acts on V^1 trivially.

Example 4.2.6 (cell attachment). We shall give an explicit model of cell attachment which is a natural generalization of [17, Prop.13.12]. Let X be a pointed connected CW complex. Let $\pi_i := \pi_i(X)$ ($i \geq 1$), $\mathcal{M}_{\text{red}}(X) = \mathcal{M} = \bigwedge(V^i, d_{\mathcal{M}})$ and $n \geq 2$. Take $a \in \pi_n(X)$. We also denote by $a : V^n \rightarrow k$ the corresponding image by the map $\pi_n(X) \rightarrow [\text{TPL}(X), \text{TPL}(S^n)] \cong (V^n)^{\vee}$ (see Lem.3.4.6). Let $X \cup_a D^{n+1}$ be the space obtained by attaching a $n+1$ -cell to X along a . A model of $X \cup_a D^{n+1}$ (i.e. a π_1^{red} -dga quasi-isomorphic to $A_{\text{red}}(X \cup_a D^{n+1})$), $\bigwedge(V^i) \oplus_a \mathcal{O}(\pi_1^{\text{red}})_l u$ is given as follows.

- As a graded module, it is $\bigwedge(V^i) \oplus \mathcal{O}(\pi_1^{\text{red}})_l u$, where $\mathcal{O}(\pi_1^{\text{red}})_l u$ is a copy of $\mathcal{O}(\pi_1^{\text{red}})_l$ whose degree is $n+1$.

2. The algebra structure is determined by that of \mathcal{M} and $\mathcal{M} \cdot (\mathcal{O}(\pi_1^{\text{red}})_l u) = (\mathcal{O}(\pi_1^{\text{red}})_l u)^2 = 0$

3. The differential d is determined by the derivation property from the formula

$$dx = \begin{cases} 0 & \text{if } x \in \mathcal{O}(\pi_1^{\text{red}})_l u \\ d_{\mathcal{M}}x, & \text{if } x \in V^i, i \neq n \\ d_{\mathcal{M}}x + a_*(x)u & \text{if } x \in V^n, \end{cases}$$

where $a_* : V^n \rightarrow \mathcal{O}(\pi_1^{\text{red}})_l$ is the composition: $V^n \xrightarrow{\text{coaction of } V^n} V^n \otimes \mathcal{O}(\pi_1^{\text{red}})_l \xrightarrow{a_* \otimes \text{id}} \mathcal{O}(\pi_1^{\text{red}})_l$.

In fact, as $\pi_1(X \cup_a D^{n+1}) \cong \pi_1(X)$, $A_{\text{red}}(X \cup_a D^{n+1})$ is isomorphic to $A_{\text{red}}(X) \times_{A_{PL}(S^n) \otimes \mathcal{O}(\pi_1^{\text{red}})_l} A_{PL}(D^{n+1}) \otimes \mathcal{O}(\pi_1^{\text{red}})_l$. So by an argument similar to [17, Prop.13.12] we see the latter is quasi-isomorphic to the above model (see also Lem.3.3.7).

We shall present some concrete examples. Suppose k is algebraically closed.

1) Let $X_0 = S^1 \times S^2$ and a_0 be a generator of $\pi_3(X_0) \cong \mathbb{Z}$. Let $X_1 = X_0 \cup_{a_0} D^4$. Using the above model, we can compute the minimal model $\mathcal{M}_1 = \mathcal{M}_{\text{red}}(X_1)$. Note that $\mathcal{O}(\mathbb{Z}^{\text{red}})$ is isomorphic to the group ring $k\langle k^* \rangle$ of the discrete group $k^* = k - \{0\}$. the fifth stage of \mathcal{M}_1 is presented as

$$\begin{aligned} \mathcal{M}_1(5) &= \bigwedge (t, s, v_{\alpha, i}, w_{\alpha})_{\alpha \in k^*, i \geq 1}, \deg t = 1, \deg s = 2, \deg v_{\alpha, i} = 4, \text{ and } \deg w_{\alpha} = 5 \\ \text{with } dv_{\alpha, 1} &= \begin{cases} ts^2 & \text{for } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}, \quad dw_{\alpha} = \begin{cases} t^3 & \text{for } \alpha = 1 \\ tv_{\alpha, 1} & \text{otherwise} \end{cases} \\ dv_{\alpha, i} &= tv_{\alpha, i-1} \text{ for all } \alpha \in k^* \text{ and } i \geq 2 \end{aligned}$$

Here, the fixed generator of $\pi_1 \cong \mathbb{Z}$ acts as $v_{\alpha, i} \mapsto \alpha v_{\alpha, i}$ and $w_{\alpha} \mapsto \alpha w_{\alpha}$. In particular, we see the fourth k -invariant of X_1 is non-zero. Independently, it is easy to see $\pi_4(X_1 \cup_{a_1} D^{n+1})$ is the free π_1 -module generated by one element, and the pro-finite dimensional completion of this has the form $(\bigoplus_{\alpha \in k^*} U_{\alpha})^{\vee}$ where π_1 acts on the infinite dimensional vector space $U_{\alpha} = k\langle u_{\alpha, 1}, u_{\alpha, 2}, \dots \rangle$ by the infinite size Jordan block of eigen-value α .

2) Next, we take an element $a_1 \in \pi_4(X_1)$ and put $X_2 = X_1 \cup_{a_1} D^5$. We can identify $\pi_4(X_1)$ with the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$, where the action of the fixed generator of π_1 corresponds the multiplication of x . we regard a_1 as a Laurent polynomial $P(x)$. Let $\phi : \pi_4(X_1) \rightarrow (\bigoplus_{\alpha \in k^*} U_{\alpha})^{\vee}$ be the structure map of the completion and put $\phi_{\alpha, i} = \phi(1)(u_{\alpha, i})$ where $1 \in \mathbb{Z}[x, x^{-1}]$. The differential of $\mathcal{M}_1(5) \oplus_{a_1} \mathcal{O}(\pi_1^{\text{red}})u$ has the following expression.

$$d\left(\sum_{i=1}^N c_i v_{\alpha, i}\right) = \delta_{\alpha, 1} c_1 ts^2 + \sum_{i=2}^N c_i tv_{\alpha, i-1} + (\phi_{\alpha, 1}, \dots, \phi_{\alpha, N}) P(A_{\alpha, N}^{-1}) \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \cdot \underline{\alpha} u.$$

Here, $\delta_{\alpha, 1}$ is the Kronecker delta, $A_{\alpha, N}$ is the Jordan block of size N and $\underline{\alpha}$ is the element of $k\langle k^* \rangle$ corresponding to α . So if we let R be the set of non-zero distinct roots of $P(x)$ and p_{α} be the multiplicity of $\alpha \in R$, $\mathcal{M}_2(5) = \bigwedge (t, s, v_{\alpha, i}, w_{\beta})_{\alpha^{-1} \in R, 1 \leq i \leq p_{\alpha}, \beta^{-1} \in R \cup \{1\}}$ with the degree and differential given by the same formula as $\mathcal{M}_1(5)$. It is easy to see $\pi_4(X_2) \cong \mathbb{Z}[x, x^{-1}]/(P(x))$ (finite rank) so we can say the profinite dimensional completion of $\pi_5(X_2)$ is the dual of $k\langle w_{\beta} \rangle_{\beta^{-1} \in R \cup \{1\}}$ (Thm.4.1.7).

Lemma 4.2.7. Let Γ be a commutative divisible group. Then the \mathbb{Q} -pro-algebraic completion $\Gamma_{\mathbb{Q}}^{\text{alg}}$ is pro-unipotent. In particular, if L is a pointed connected simplicial set with $\pi_1(L)$ commutative and divisible, $\mathcal{M}_{\text{red}}(L; \mathbb{Q}) \cong \mathcal{M}_{PL}(L; \mathbb{Q})$.

Proof. Left to the reader. □

Theorem 4.2.8. Let $0 \rightarrow M \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1$ be an extension of groups such that M is a finitely generated abelian group or a finite dimensional \mathbb{Q} -vector space. We regard M as a Γ -module with the action induced by the extension. Then

$$\mathcal{M}_{\text{red}}(K(\tilde{\Gamma}, 1); k) \cong \mathcal{M}_{\text{red}}(K(\Gamma, 1); k) \otimes_{(\alpha, \eta)} \bigwedge [((M \otimes k^{\text{ss}}))^{\vee}, 1].$$

See Def.3.3.8 for the notation. Here, η is the MC-element corresponding to the Γ -module $M \otimes k$ and $\alpha \in Z^2 T \mathcal{M}_{\text{red}}(K(\Gamma, 1))((M \otimes k)^{\text{ss}}, \eta)$ is a cocycle which represents the class $[\tilde{\Gamma}] \otimes k \in H^2(\Gamma, M \otimes k)$. In particular, if Γ is algebraically good over k , $\tilde{\Gamma}$ is also algebraically good over k . (In these statements, when M satisfies the second condition, we assume $k = \mathbb{Q}$.)

Proof. When we use Thm.4.2.3 or Lem.4.2.7, the proof is similar to that of Thm.4.2.3. \square

4.2.1 components of free loop spaces

Let K be a connected fibrant simplicial set and π_1 be the fundamental group of K with respect to a fixed base point $*$. Let ΛK be the free loop space of K , i.e., the internal hom-object $\mathfrak{Hom}(S^1, K)$ in the category of unpointed simplicial sets. Choose a point $b \in S^1$. The evaluation at b defines a fibration $p : \Lambda K \rightarrow K$ whose fiber is the pointed loop space ΩK of K . Let $\gamma \in \pi_1$. We let γ denote its representative loop and $\Lambda_\gamma K$ do the connected component of ΛK containing γ . p induces a fiber sequence $\Omega_\gamma K \rightarrow \Lambda_\gamma K \rightarrow K$. Let $\partial : \pi_i(K) \rightarrow \pi_{i-1}(\Omega_\gamma K) \cong \pi_i(K)$ be the boundary map of the long exact sequence of this fiber sequence. One can easily see for $\alpha \in \pi_i(K)$, $\partial(\alpha)$ vanishes if and only if the Whitehead product $[\gamma, \alpha]$ vanishes. So there exist exact sequences of groups

$$\begin{aligned} 0 &\rightarrow \pi_2(K)_\gamma \rightarrow \pi_1(\Lambda_\gamma K) \rightarrow C_{\pi_1}(\gamma) \rightarrow 1, \\ 0 &\rightarrow \pi_{i+1}(K)_\gamma \rightarrow \pi_i(\Lambda_\gamma K) \rightarrow \pi_i(K)^\gamma \rightarrow 0 \end{aligned}$$

for $i \geq 2$, where $\pi_i(K)^\gamma = \text{Ker}(\text{id} - \gamma)$, $\pi_i(K)_\gamma = \text{Coker}(\text{id} - \gamma)$ (γ means the action on the homotopy groups), and $C_{\pi_1}(\gamma)$ denotes the centralizer of γ in π_1 . Thus, as is well-known, different components of ΛK have different homotopy types. In this sub-subsection, we give a model of $\Lambda_\gamma K$ under the assumption that γ is in the center of π_1 . For the nilpotent case, the result here is already included in [13] and we use their argument in the proof. The difference form the nilpotent one is that the category of unpointed schematic homotopy types is not equivalent to that of unaugmented equivariant dga's.

Let $\mathcal{M} = \mathcal{M}_{\text{red}}(K, *)$. Let $\Lambda \mathcal{M}$ be a π_1^{red} -dga defined as follows.

1. $\Lambda \mathcal{M} = \bigwedge(V^i, \bar{V}^i) = \bigwedge(V^i) \otimes \bigwedge(\bar{V}^i)$, where for $i \geq 1$, \bar{V}^i is a copy of V^i with $\deg \bar{V}^i = i - 1$. We identify \mathcal{M} with a subalgebra of $\Lambda \mathcal{M}$ via the natural identification : $\mathcal{M} \supset V^i = V^i \subset \Lambda \mathcal{M}$.
2. We define a derivation $i : \mathcal{M} \rightarrow \Lambda \mathcal{M}$ of degree -1 by $i(x) = \bar{x}$ ($x \in V^i$, $\bar{x} \in \bar{V}^i$ is the copy of x), $i(xy) = i(x)y + (-1)^{\deg x} xi(y)$. Then, the differential $d_{\Lambda \mathcal{M}}$ on $\Lambda \mathcal{M}$ is defined by $i \circ d_{\mathcal{M}} + d_{\Lambda \mathcal{M}} \circ i = 0$.

Suppose $\gamma \in \pi_1$ be in the center of π_1 . Let $\bar{\gamma}$ be the image of γ under the map $\pi_1 \rightarrow \pi_1^{\text{alg}}(k)$ and $\bar{\gamma} = (u, s)$ be the decomposition such that $u \in R_u(\pi_1^{\text{alg}}(k))$, $s \in \pi_1^{\text{red}}(k)$. Under the identification of Cor.5.2.6, we regard u as a linear map $\bar{V}^1 \cong V^1 \rightarrow k$. Let $I \subset \Lambda \mathcal{M}$ be the homogeneous ideal generated by

$$\{x - u(x), dx \mid x \in \bar{V}^1\} \cup \{y - s \cdot y \mid y \in \Lambda \mathcal{M}\}.$$

This ideal is closed under π_1^{red} -action and differential. The first part of the above set is the same as K_u in [13, P.4946]. We define a $\pi_1(\Lambda_\gamma K, \gamma)^{\text{red}}$ -dga $\Lambda_\gamma \mathcal{M}$ by

$$\Lambda_\gamma \mathcal{M} := \Lambda \mathcal{M}/I,$$

where the action of $\pi_1(\Lambda_\gamma K, \gamma)^{\text{red}}$ is given by the pullback of the action of π_1 by the evaluation $\Lambda_\gamma K \rightarrow K$ at some fixed point of S^1 . $\Lambda_\gamma \mathcal{M}$ is not necessarily minimal. The following is a non-simply connected version of [7, Theorem].

Proposition 4.2.9. *Suppose $k = \mathbb{Q}$. Let K be a connected simplicial set such that $\pi_1(K)$ is algebraically good, $\pi_2(K)$ is a finitely generated abelian group or finite dimensional \mathbb{Q} -vector space, and $\pi_i(K)$ is of finite rank for each $i \geq 3$. Let γ be an element of the center of $\pi_1(K)$. Then, under the above notations, there exists a quasi-isomorphism $\mathcal{M}_{\text{red}}(\Lambda_\gamma K, \gamma) \xrightarrow{\sim} \Lambda_\gamma \mathcal{M}$.*

Proof. See also Rem.4.2.10 below. In the following, $(-\otimes\mathbb{Q})^{\text{schn}}$ is abbreviated to $(-)^{\text{schn}}$. Let $(K^{\text{schn}})^{S^1} \in \text{sPr}(\mathbb{Q})$ be an object given by $(K^{\text{schn}})^{S^1}(R) = (K^{\text{schn}}(R))^{S^1}$ for $R \in \mathbb{Q}-\text{Alg}$, where the right hand side is the exponential in sSet and K^{schn} is taken to be fibrant in $\text{sPr}(\mathbb{Q})^{\text{loc}}$. Let $(K^{\text{schn}})_\gamma^{S^1}$ be the connected component of $(K^{\text{schn}})^{S^1}$ containing γ . In general, we have a map $f : (\Lambda_\gamma K)^{\text{schn}} \rightarrow (K^{\text{schn}})_\gamma^{S^1}$ induced by the map $K \rightarrow K^{\text{schn}}(k)$.

We first consider the case that $V = \pi_2(K)$ is a \mathbb{Q} -vector space. In this case, as $V^{\text{alg}} \cong V$ by Lem.4.2.7, by comparing the long exact sequences associated to the fibrations $p^{\text{schn}} : (\Lambda_\gamma K)^{\text{schn}} \rightarrow K^{\text{schn}}$, $q : (K^{\text{schn}})_\gamma^{S^1} \rightarrow K^{\text{schn}}$, the evaluation at the base point of S^1 , we can see f is a weak equivalence and $\pi_1(\Lambda_\gamma K)^{\text{red}} \cong \pi_1^{\text{red}}$. Let $G = \pi_1^{\text{red}}$. Let $(H, B, a_B) \in \text{dgAlg}_{0,*}^{\text{red}}$ be a reductive dga and $\psi : H \rightarrow G$ be a homomorphism. We fix three sets H_1 , H_2 , and H_3 as follows.

$$\begin{aligned} H_1 &= \{\beta \in [\psi'^* \mathcal{M}, \wedge \xi \otimes B]_{\text{dgAlg}(\mathbb{Z}^{\text{red}} \times H)} \mid a_B \circ \beta = u \in [s^* \mathcal{M}, \wedge \xi]_{\text{dgAlg}(\mathbb{Z}^{\text{red}})}\} \\ H_2 &= \{\beta' \in [\psi'^* \mathcal{M}, \wedge \xi \otimes B]'_{\text{dgAlg}(\mathbb{Z}^{\text{red}} \times H)} \mid a_B \circ \beta' = u \in [s^* \mathcal{M}, \wedge \xi]'_{\text{dgAlg}(\mathbb{Z}^{\text{red}})}\} \\ H_3 &= \left\{ \beta' \in [\psi'^* \mathcal{M}, \wedge \xi \otimes B]'_{\text{dgAlg}(\mathbb{Z}^{\text{red}} \times H)} \middle| \begin{array}{l} \text{for a representative } \tilde{\beta}' \in [\psi'^* \mathcal{M}, \wedge \xi \otimes B] \text{ of } \beta', \\ a_B \circ \tilde{\beta}' = u \in [s^* \mathcal{M}, \wedge \xi] \end{array} \right\} \end{aligned}$$

For the notation $[-, -]',$ see Lem.3.3.7 and below. (we put $E = *$). Here, $(\mathbb{Z}^{\text{red}}, \wedge \xi)$ is the minimal model of $A_{\text{red}}(S^1)$ and ξ denotes a generator of degree 1, $\psi' = s \times \psi$, and u is considered as a morphism $s^* \mathcal{M} \rightarrow \wedge \xi \in \text{dgAlg}(\mathbb{Z}^{\text{red}})$. By an argument similar to the proof of [13, Thm.6.1], we see that there exists a natural bijection:

$$[\psi^* \Lambda_\gamma \mathcal{M}, B]_{\text{dgAlg}(H)} \cong H_1 \tag{*}$$

(Here we use H -equivariant affine stacks[25] instead of rational simplicial sets in [13]. and note that there is a natural bijection

$$[A, U(C)]_{\text{dgAlg}(H \times \mathbb{Z}^{\text{red}})} \cong [A_{\mathbb{Z}}, C]_{\text{dgAlg}(H)}$$

,where for an H -dga A , and an $H \times \mathbb{Z}^{\text{red}}$ -dga C , $A_{\mathbb{Z}}$ denotes \mathbb{Z} -coinvariants of A , and $U(C)$ is C considered as $H \times \mathbb{Z}^{\text{red}}$ -dga with trivial \mathbb{Z}^{red} -action.)

We shall see what universal property $\mathcal{M}_{\text{red}}((K^{\text{schn}})_\gamma^{S^1})$ have, using Lem.3.3.7. We work on the intermediate category whose objects are pointed schematic homotopy types but whose morphisms are those of $\text{Ho}(\text{SHT})$, the unpointed homotopy category between underlying unpointed schematic homotopy types. $(K^{\text{schn}})_\gamma^{S^1}$ have a universal property as follows. Let (Y, y) be a pointed schematic homotopy type. The following two sets of morphisms in $\text{Ho}(\text{SHT})$ is naturally bijective.

1. morphisms $\phi : Y \rightarrow (K^{\text{schn}})_\gamma^{S^1} \in \text{Ho}(\text{SHT})$ such that $\pi_0(\phi(\mathbb{Q})) : \pi_0(Y(\mathbb{Q})) \rightarrow \pi_0((K^{\text{schn}})_\gamma^{S^1}(\mathbb{Q}))$ maps $[y]$ to $[\gamma]$.
2. morphisms $\varphi : Y \times (S^1)^{\text{schn}} \rightarrow K^{\text{schn}}$ such that the composition $y \times S^1 \rightarrow Y \times (S^1)^{\text{schn}}(\mathbb{Q}) \rightarrow K^{\text{schn}}(\mathbb{Q})$ is freely homotopic to γ .

As γ is in the center, this bijection gives the following bijection.

$$\begin{aligned} \{\alpha \in [\text{Rep}(G)^c, \text{Rep}(H')] \mid \omega_{H'} \circ \alpha = \omega_G \in [\text{Rep}(G)^c, \text{Vect}]\} \\ \cong \{\alpha' \in [\text{Rep}(G)^c, \text{Rep}(H' \times \mathbb{Z}^{\text{alg}})] \mid i_{\mathbb{Z}}^* \circ \alpha' = s \in [\text{Rep}(G)^c, \text{Rep}(\mathbb{Z}^{\text{alg}})]\}, \end{aligned} \quad (**)$$

see Lem.3.3.7. Here $[-, -]$ means $[-, -]_{\text{Cat}^{\text{cl}}}$ so " $=$ " means naturally isomorphic, respecting tensors, and in the left hand side G is identified with $\pi_1(\Lambda_{\gamma}K)^{\text{red}}$ while it is identified with $\pi_1(K)^{\text{red}}$ in the right hand side ($H' = \pi_1(Y)$). So by Lem.3.3.7 and the universal property of $(K^{\text{sch}})_{\gamma}^{S^1}$, we get a natural bijection:

$$[\psi^* \mathcal{M}_{\text{red}}((K^{\text{sch}})_{\gamma}^{S^1}), B]_{\text{dgAlg}(H)}' \cong H_2.$$

On the other hand, $(*)$ implies $[\psi^* \Lambda_{\gamma} \mathcal{M}, B]_{\text{dgAlg}(H)}' \cong H_3$. Now, elements in $[s\mathcal{M}, \wedge \xi]$ which is identified with u in $[s\mathcal{M}, \wedge \xi]'$ are $\{u * g \mid g \in G(\mathbb{Q})\}$ but by assumption these are equal to u . So $H_2 = H_3$, which implies $\Lambda_{\gamma} \mathcal{M} \simeq \mathcal{M}_{\text{red}}((K^{\text{sch}})_{\gamma}^{S^1})$.

For the case $\pi_2(K)$ finitely generated abelian, f is not a weak equivalence. Let $r : K \rightarrow K_{\mathbb{Q}}$ be a fiberwise rationalization so that $\pi_1(r) : \pi_1(K) \cong \pi_1(K_{\mathbb{Q}})$ and $\pi_i(r) \otimes \mathbb{Q} : \pi_i(K) \otimes \mathbb{Q} \cong \pi_i(K_{\mathbb{Q}})$. Consider the induced map $r_* : \Lambda_{\gamma} K \rightarrow \Lambda_{\gamma} K_{\mathbb{Q}}$. By Lem.4.2.7, the pull-back by $\pi_1(r_*)$ preserves semi-simple \mathbb{Q} -representations so maps $\pi_1(r_*)^{\text{red}} : \pi_1(\Lambda_{\gamma} K)^{\text{red}} \rightarrow \pi_1(\Lambda_{\gamma} K_{\mathbb{Q}})^{\text{red}}$ and $A_{\text{red}}(r_*) : A_{\text{red}}(K_{\mathbb{Q}}) \rightarrow A_{\text{red}}(K)$ are induced. By 3.3.7, $(\alpha_*)^* TPL(K_{\mathbb{Q}}) \rightarrow TPL(K)$ is equivalent to $T A_{\text{red}}(\alpha_*) : T A_{\text{red}}(K_{\mathbb{Q}}) \rightarrow T A_{\text{red}}(K)$. So by Thm.4.1.3, Thm.4.2.8 and naturality of the construction of the iterated Hirsch extensions, we see $\mathcal{M}_{\text{red}}(\Lambda_{\gamma} K_{\mathbb{Q}}) \cong \mathcal{M}_{\text{red}}(\Lambda_{\gamma} K)$ as underlying dga's. \square

Remark 4.2.10. If we define a "category of dg-algebras over pro-reductive groupoids" appropriately, it is equivalent to the sub-category of $\text{Ho}(\text{SHT})$ whose morphisms are those which preserve semisimple local systems. We can prove Prop.4.2.9, using this category similarly. To deal with more general base loops than the center, especially loops which do not preserve semisimple local systems, it will be inevitable to consider Tannakian dgc's as the universality of the component is not contained in the semisimple subcategory.

Example 4.2.11. Let K be a connected simplicial set such that $\pi_1(K) = \mathbb{Z}$ and $\mathcal{M} (= \mathcal{M}_{\text{red}}(K)) = \wedge(t, s_1, s_2, s_3, s_4, u_1, u_2, u_3)$ $\deg t = 1, \deg s_i = 2, \deg u_j = 3$, where

1. the generator $g = 1 \in \mathbb{Z}$ act on \mathcal{M} by $g \cdot t = t, g \cdot (s_i, s_{i+1}) = (s_i, s_{i+1}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ($i = 1, 3$), $g \cdot u_j = -u_j$ ($j = 1, 2$), $g \cdot u_3 = u_3$, and
2. the differential is given by $ds_1 = ds_2 = 0, ds_3 = ts_1, ds_4 = ts_2, du_1 = s_1 s_2, du_2 = 2tu_1 - s_1 s_3 - s_2 s_4, du_3 = s_1 s_4 - s_2 s_3$.

For example, the action of π_1 on $\pi_2 \otimes \mathbb{Q}$ ($\pi_i = \pi_i(K)$) is as follows (see the proof of Thm.3.4.5).

$$g \cdot (s_4^*, s_3^*, s_2^*, s_1^*) = (s_4^*, s_3^*, s_2^*, s_1^*) \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Here, (s_i^*) is the dual basis of (s_i) .

(i) $\gamma = e$ (the unity). $\Lambda_e \mathcal{M} = \wedge(t, \bar{s}_i, s_i, \bar{u}_j, u_j)_{i=1, \dots, 4, j=1, 2, 3}$, which is already minimal. For example, $d\bar{u}_2 = 2t\bar{u}_1 + \bar{s}_1 s_3 + s_1 \bar{s}_3 + \bar{s}_2 s_3 + s_2 \bar{s}_3$. In this case, $\pi_1(\Lambda_e K) \cong \pi_2 \rtimes \pi_1$ and by the description of the

minimal model, we can compute the action of $\pi_1(\Lambda_e K)$ on the homotopy group. For example, the action of \bar{s}_1 (considered as an element of $\pi_2 \subset \pi_2 \times \pi_1$) on $\pi_2(\Lambda_e K) \otimes \mathbb{Q} \cong \mathbb{Q}\langle \bar{u}_3^*, \bar{u}_2^*, \bar{u}_1^*, s_4^*, s_3^*, s_2^*, s_1^* \rangle$ is given by

$$\begin{pmatrix} 1 & & 1 & & \\ & 1 & & -1 & \\ & & 1 & & 1 \\ & & & 1 & \\ & & & & 1 \\ & & & & & 1 \end{pmatrix}$$

- (ii) $\gamma = g$. $\Lambda_g \mathcal{M} = \wedge(t, \bar{u}_3, u_3)$ with $d = 0$, which is minimal
- (iii) $\gamma = g^2$. $\Lambda_{g^2} \mathcal{M}$ is not minimal. A minimal model of $\Lambda_{g^2} \mathcal{M}$ is $\mathcal{N}_2 := \wedge(t, \bar{u}_1, \bar{u}_3, u'_2, u_3)$, $\deg t = 1$, $\deg \bar{u}_1 = \deg \bar{u}_3 = 2$, $\deg u'_2 = \deg u_3 = 3$, $d = 0$. \mathcal{N}_2 is identified with a subalgebra of $\Lambda_{g^2} \mathcal{M}$ whose inclusion is a quasi-isomorphism, by $t \mapsto t$, $\bar{u}_j \mapsto \bar{u}_j$, $u'_2 \mapsto u_2 - t\bar{u}_2/2$, $u_3 \mapsto u_3$.
- (iv) $\gamma = g^4$. A minimal model of $\Lambda_{g^4} \mathcal{M}$ is $\mathcal{N}_4 := \wedge(t, \bar{s}_1, \bar{s}_2, s'_3, s'_4, \bar{u}'_3, \bar{u}'_2, u'_3)$ with $\deg s'_i = \deg \bar{u}'_j = 2$, $\deg u'_j = 3$, and $ds'_3 = ds'_4 = d\bar{u}'_1 = 0$, $d\bar{u}'_3 = -\bar{s}_1 s'_4 + \bar{s}_2 s'_3$, $du'_2 = du'_3 = 0$. \mathcal{N}_4 is identified with a subalgebra of $\Lambda_{g^4} \mathcal{M}$ whose inclusion is a quasi-isomorphism, by $x \mapsto x$ ($x = t, \bar{s}_1, \bar{s}_2$), $s'_i \mapsto s_i - t\bar{s}_i/4$ ($i = 3, 4$), $\bar{u}'_1 \mapsto \bar{u}_1 + (\bar{s}_1 \bar{s}_4 - \bar{s}_3 \bar{s}_2)/4$, $\bar{u}'_3 \mapsto \bar{u}_3 - \bar{s}_3 \bar{s}_4/4$, $u'_2 \mapsto u_2 - (\bar{s}_3 s'_3 + \bar{s}_4 s'_4 + t\bar{u}_2)/4$. $u'_3 \mapsto u_3 - t\bar{u}_3/4 - 13t\bar{s}_3 \bar{s}_4/16 + (\bar{s}_3 s_4 - \bar{s}_4 s_3)/4$.

Remark 4.2.12. Let $[\gamma] \in H^2(\pi_1; (\pi_2)_\gamma)$ be the class associated to the extension $0 \rightarrow (\pi_2)_\gamma \rightarrow \pi_1(\Lambda_\gamma K) \rightarrow \pi_1 \rightarrow 1$. When γ is the unity, clearly $[\gamma] = 0$. In the above example, for all γ , the class $[\gamma] \otimes \mathbb{Q} \in H^2(\pi_1; (\pi_2)_\gamma \otimes \mathbb{Q})$ is zero. But this is not true in general, and a counterexample exists even in the nilpotent case. For example, let $\mathcal{M} = \wedge(t_1, t_2, t_3, s)$ with $\deg t_i = 1$, $\deg s = 2$, $dt_i = 0$, and $ds = t_1 t_2 t_3$, and let $\gamma = (t_1 = 1, t_2 = 0, t_3 = 0)$. Then $\Lambda_\gamma \mathcal{M} = \wedge(t_1, t_2, t_3, \bar{s}, s)$ with $d\bar{s} = -t_2 t_3$, which imply $[\gamma] \otimes \mathbb{Q} = [-t_2 t_3] \neq 0$ (see Thm.4.2.8).

4.3 Equivalence with algebraically good spaces

In this subsection, we will show a homotopy category of Tannakian dgc's with subsidiary data is equivalent to a homotopy category of some spaces. We restrict our discussion to the pointed case. Let $sSet_*^c$ denote the category of pointed connected simplicial sets. We say a connected pointed simplicial set K is *algebraically good* if $\pi_1(K)$ is algebraically good and $\pi_i(K)$ is a finite dimensional \mathbb{Q} -vector space for each $i \geq 2$. we denote by $sSet_*^{gd}$ the full subcategory of $sSet_*^c$ consisting of algebraically good spaces.

Definition 4.3.1. (1) Define a category Tan_*^+ as follows.

1. An object is a triple (T, Γ, ϕ) consisting of a Tannakian dgc T , a discrete group Γ and an equivalence of closed k -categories $\phi : Z^0 T \xrightarrow{\sim} \text{Rep}(\Gamma)$ such that the following diagram is commutative.

$$\begin{array}{ccc} Z^0 T & \xrightarrow{\phi} & \text{Rep}(\Gamma) \\ & \searrow Z^0 \omega_T & \downarrow \omega_\Gamma \\ & & \text{Vect}, \end{array}$$

where ω_Γ is the forgetful functor.

2. A morphism $(T, \Gamma, \phi) \rightarrow (T', \Gamma', \phi')$ is a pair $F = (F, F^{gr})$ of a morphism of Tannakian dgc's $F : T \rightarrow T'$ and a group homomorphism $F^{gr} : \Gamma' \rightarrow \Gamma$ such that the following diagram is commutative.

$$\begin{array}{ccc} Z^0 T & \xrightarrow{Z^0 F} & Z^0 T' \\ \downarrow \phi & & \downarrow \phi' \\ \text{Rep}(\Gamma) & \xrightarrow{(F^{gr})^*} & \text{Rep}(\Gamma') \end{array}$$

We say a morphism $F : (T, \Gamma, \phi) \rightarrow (T', \Gamma', \phi') \in \text{Tan}_*^+$ is a weak equivalence if $F : T \rightarrow T'$ is a quasi-equivalence and F^{gr} is an isomorphism. $\text{Ho}(\text{Tan}_*^+)$ denotes the localization of Tan_*^+ obtained by inverting all weak equivalences.

(2) Tan_*^{+gd} denotes the full subcategory consisting of (T, Γ, ϕ) 's such that $\pi_i(T) := [T, \text{TPL}S^i]_{\text{dgCat}_*^{\text{cl}}}$ is a finite dimensional \mathbb{Q} -vector space for $i \geq 2$ and Γ is algebraically good (By Lem.3.4.6, $\pi_i(T)$ is isomorphic to the dual of the i -th indecomposable module of the minimal model of the corresponding dga). The corresponding full subcategory of $\text{Ho}(\text{Tan}_*^+)$ is denoted by $\text{Ho}(\text{Tan}_*^{+gd})$. (3) Let $F_0, F_1 : (T, \Gamma, \phi) \rightarrow (T', \Gamma', \phi') \in \text{Tan}_*^+$ be two morphisms. We say F_1 and F_2 are right homotopic (resp. left homotopic), written \sim_r (resp. \sim_l) if $F_1^{gr} = F_2^{gr}$ and F_0 and $F_1 : T \rightarrow T'$ are right homotopic (resp. left homotopic) as morphisms of $\text{dgCat}_*^{\text{cl}}$.

We shall show $\text{Ho}(\text{sSet}_*^{gd})$ and $\text{Ho}(\text{Tan}_*^{+gd})$ are equivalent.

We do not claim Tan_*^+ has a model category structure but we have the following lemma.

Lemma 4.3.2. Let $(T, \Gamma, \phi), (T', \Gamma', \phi') \in \text{Tan}_*^+$. Suppose T is cofibrant and T' is fibrant in $\text{dgCat}_*^{\text{cl}}$. Then, for two morphism $F_0, F_1 : (T, \Gamma, \phi) \rightarrow (T', \Gamma', \phi') \in \text{Tan}_*^+$, $F_0 \sim_r F_1$ if and only if $F_0 \sim_l F_1$. \sim_r (\sim_l) is an equivalence relation on $\text{Hom}_{\text{Tan}_*^+}((T, \Gamma, \phi), (T', \Gamma', \phi'))$ and there exists a natural bijection:

$$\text{Hom}_{\text{Ho}(\text{Tan}_*^+)}((T, \Gamma, \phi), (T', \Gamma', \phi')) \cong \text{Hom}_{\text{Tan}_*^+}((T, \Gamma, \phi), (T', \Gamma', \phi')) / \sim_r.$$

Proof. Note that if $F_0, F_1 : T \rightarrow T' \in \text{Tan}_*$ are right or left homotopic in $\text{dgCat}_*^{\text{cl}}$, the induced morphisms $(Z^0 F_0)^*, (Z^0 F_1)^* : \text{Aut}^\otimes(\omega_{Z^0 T'}) \rightarrow \text{Aut}^\otimes(\omega_{Z^0 T})$ are equal. By using this fact, we see right or left homotopic morphisms represent the same morphism in $\text{Ho}(\text{Tan}_*^+)$. So the lemma is standard, see [1] or [14, Thm.1.2.10]. \square

Remark 4.3.3. The author does not know whether the statement corresponding to Lem.4.3.2 in the unpointed case is true. In this case, we will consider groupoids instead of groups. As morphisms of groupoids have non-trivial homotopic relation, if we define a right or left homotopic relation similarly, the proof does not go on similary.

Define two functors

$$\text{T}_{\text{PL}}^+ : \text{sSet}_*^c \longrightarrow (\text{Tan}_*^+)^{\text{op}}, \quad \mathbb{R}\langle -, -, - \rangle : (\text{Tan}_*^+)^{\text{op}} \longrightarrow \text{sSet}_*^c$$

as follows.

For $K \in \text{sSet}_*^c$, $\text{T}_{\text{PL}}^+(K) = (\text{T}_{\text{PL}}(K), \pi_1(K), \phi_K)$, where ϕ_K is a fixed functorial equivalence $Z^0(\text{T}_{\text{PL}}(K)) \xrightarrow{\sim} \text{Loc}(K) \xrightarrow{\sim} \text{Rep}(\pi_1(K))$. For $(T, \Gamma, \phi) \in \text{Tan}_*^+$, $\mathbb{R}\langle T, \Gamma, \phi \rangle$ is the following pullback.

$$\begin{array}{ccc} \mathbb{R}\langle T, \Gamma, \phi \rangle & \longrightarrow & \langle QT \rangle \\ \downarrow & & \downarrow p \\ B\Gamma & \xrightarrow{\phi^*} & B\Pi_1\langle QT \rangle \end{array}$$

Here, QT is a fixed functorial cofibrant replacement of T , $\Pi_1(QT)$ is the fundamental groupoid, p is the canonical map which gives an isomorphism of fundamental groupoid, and ϕ^* is the following composition.

$$B\Gamma \rightarrow B\text{Aut}^\otimes(\omega_\Gamma) \xrightarrow{\phi^*} B\text{Aut}^\otimes(Z^0\omega_{QT}) \cong B\pi_1(QT) \rightarrow B\Pi_1(T),$$

see Appendix A.2, where the first map is induced by the natural "evaluation" $\Gamma \rightarrow \text{Aut}^\otimes(\omega_\Gamma)$ and the inverse of the isomorphism $\text{Aut}^\otimes(Z^0\omega_T) \cong \pi_1(QT)$ is the following composition:

$$\pi_1(QT) \cong [QT, T_{\text{PL}} S^1]_{\text{dgCat}_*^{\text{cl}}} \xrightarrow{Z^0} [Z^0(QT), \text{Rep}(\mathbb{Z})]_{\text{Cat}_*^{\text{cl}}} \cong \text{Aut}^\otimes(Z^0\omega_{QT}),$$

where the last isomorphism sends an action of $1 \in \mathbb{Z}$ to an automorphism. This is in fact, an isomorphism by Thm.?? and 5.2.1. Note that T_{PL}^+ preserves weak equivalences so it induces a functor $T_{\text{PL}}^+ : \text{Ho}(\text{sSet}_*^c) \rightarrow \text{Ho}(\text{Tan}_*^+)^{\text{op}}$. As the map $p : \langle QT \rangle \rightarrow B\Pi_1(QT)$ is a fibration, the above pullback is a homotopy pullback. So $\mathbb{R}\langle -, -, - \rangle$ also preserves weak equivalences and it defines a functor between homotopy categories. We have an obvious natural map

$$\Phi : \text{Hom}_{\text{sSet}_*^c}(K, \mathbb{R}\langle T, \Gamma, \phi \rangle) \longrightarrow \text{Hom}_{\text{Tan}_*^+}((QT, \Gamma, \phi), T_{\text{PL}}^+(K)).$$

We can easily see if K is reduced, i.e., $K_0 \cong *$, this map is a bijection and preserves and reflects homotopy relation so Φ induces a bijection

$$\Phi : \text{Hom}_{\text{Ho}(\text{sSet}_*^c)}(K, \mathbb{R}\langle T, \Gamma, \phi \rangle) \longrightarrow \text{Hom}_{\text{Ho}(\text{Tan}_*^+)}((T, \Gamma, \phi), T_{\text{PL}}^+(K))$$

by Lem.4.3.2. Thus the pair $(T_{\text{PL}}^+, \mathbb{R}\langle -, -, - \rangle)$ is an adjoint pair.

Theorem 4.3.4. (1) *The above adjoint pair induces an equivalence between full subcategories :*

$$T_{\text{PL}}^+ : \text{Ho}(\text{sSet}_*^{gd}) \rightleftarrows \text{Ho}(\text{Tan}_*^{gd})^{\text{op}} : \mathbb{R}\langle -, -, - \rangle.$$

(2) *Let K be a connected pointed simplicial set whose fundamental group is algebraically good and whose higher homotopy groups are of finite rank as Abelian groups. Then, the unit $K \rightarrow \mathbb{R}\langle T_{\text{PL}}(K), \pi_1(K), \phi_K \rangle$ is the fiberwise rationalization.*

Proof. (1) For $K \in \text{sSet}_*^{gd}$, we shall show the unit map $u_K : K \rightarrow \mathbb{R}\langle T_{\text{PL}}^+(K) \rangle$ is a weak equivalence. we use the notation of Thm.4.1.3. By definition, the map $\pi_1(u_K) : \pi_1(K) \rightarrow \pi_1(\mathbb{R}\langle T_{\text{PL}}^+(K) \rangle)$ is an isomorphism. For i -th homotopy group ($i \geq 2$), we may assume K is weak equivalent to the i -th level of the Postnikov tower of K . Consider the fiber sequence $F \xrightarrow{f} K \xrightarrow{p_{i-1}} K_{(i-1)}$, where F is the fiber of type $K(\pi_i(K), i)$. By the proof of Lem.4.1.5, the corresponding morphism $f^* : T_{\text{PL}} K \rightarrow T_{\text{PL}} F$ is equivalent to the morphism $T(\pi_1^{\text{red}}, \mathcal{M}) \rightarrow T(e, \wedge(\pi_i^V, i)) \in \text{Tan}_*$ induced by a morphism $(\pi_1^{\text{red}}, \mathcal{M}) \rightarrow (e, \wedge(\pi_i^V, i)) \in \text{dgAlg}_{0,*}^{\text{red}}$ which maps V^j to 0 ($j < i$) and V^i to π_i^V isomorphically. So by Lem.3.4.6, $\pi_i(\mathbb{R}\langle f^* \rangle) : \pi_i(\mathbb{R}\langle T_{\text{PL}} F \rangle) \rightarrow \pi_i(\mathbb{R}\langle T_{\text{PL}} K \rangle)$ is an isomorphism. As the unit $F \rightarrow \mathbb{R}\langle T_{\text{PL}} F \rangle$ is a weak equivalence by the classical Sullivan's theory, we see the map $\pi_i(K) \rightarrow \pi_i(\mathbb{R}\langle T_{\text{PL}} K \rangle)$ is an isomorphism. This implies $\pi_i(u_K)$ is an isomorphism. Thus, u_K is a weak equivalence and T_{PL}^+ is fully faithful.

By a similar argument, we see a map $F : (T, \Gamma, \phi) \rightarrow (T', \Gamma', \phi') \in \text{Tan}_*^{gd}$ is a weak equivalence if the induced map $\mathbb{R}F : \mathbb{R}\langle T, \Gamma, \phi \rangle \rightarrow \mathbb{R}\langle T', \Gamma', \phi' \rangle$ is a weak equivalence so T_{PL}^+ is essentially surjective. One can prove (2) by a similar argument. \square

5 Correspondence with schematic homotopy types

5.1 Definitions

We recall the notion of schematic homotopy types from [22, 25]. Let $R - \text{Alg}$ denote the category of (discrete) commutative unital R -algebras for a commutative unital k -algebra R .

Definition 5.1.1. (1) $\text{sPr}(k)$ denotes the category of simplicial presheaves on $\text{Aff}_k := (k - \text{Alg})^{\text{op}}$. In other words, $\text{sPr}(k)$ is the category of functors $k - \text{Alg} \rightarrow \text{sSet}$ and natural transformations.

1. For a simplicial presheaf $X \in \text{Ob}(\text{sPr}(k))$ and a R -point $s \in X(R)_0$, ($R \in k - \text{Alg}$) the i -th homotopy presheaf $\pi_i^{\text{pre}}(X, s)$ with base point s is a presheaf on $\text{Aff}_R = (R - \text{Alg})^{\text{op}}$ defined by $\pi_i^{\text{pre}}(X, s)(S) := \pi_i(X(S), u(s))$ for $S \in R - \text{Alg}$. Here, $u(s)$ denotes the image of s by the unit $u : R \rightarrow S$, and if $i = 0$, we ignore s and suppose $R = k$. The i -th homotopy sheaf $\pi_i(X, s)$ with base point s is the sheafification of $\pi_i^{\text{pre}}(X, s)$ with respect to the faithfully flat quasi-compact topology on Aff_R . We say $X \in \text{sPr}(k)$ is connected if the 0-th homotopy sheaf $\pi_0(X)$ is one point sheaf.
2. We regard $\text{sPr}(k)$ as a model category via the objectwise projective model structure, denoted by $\text{sPr}(k)^{\text{obj}}$, where a morphism $f : X \rightarrow Y \in \text{sPr}(k)$ is a weak equivalence (resp. a fibration) if and only if $f_R : X(R) \rightarrow Y(R) \in \text{sSet}$ is a weak equivalence (resp. fibration) of simplicial sets for any $R \in k - \text{Alg}$. We call the above weak equivalence an objectwise equivalence.
3. We also consider the local projective model structure [18, 19] on $\text{sPr}(k)$, denoted by $\text{sPr}(k)^{\text{loc}}$, where weak equivalences and cofibrations are defined as follows.
 - (a) A morphism $f : X \rightarrow Y$ is a weak equivalence, called a local equivalence, if and only if it induces isomorphisms of 0-th homotopy sheaves $\pi_0(X) \cong \pi_0(Y)$ and of i -th homotopy sheaves $\pi_i(X, s) \cong \pi_i(Y, f(s))$ for any $i \geq 1$, any $R \in k - \text{Alg}$, and any base point $s \in X(R)_0$.
 - (b) A morphism is a cofibration if and only if it is a cofibration in the objectwise projective model structure.

We say an object $X \in \text{sPr}(k)$ is a local object if a fibrant replacement $X \rightarrow RX$ in $\text{sPr}(k)^{\text{loc}}$ is an objectwise equivalence.

4. We set $\text{sPr}(k)_* := */\text{sPr}(k)$, $\text{sPr}(k)_*^{\text{obj}} := */\text{sPr}(k)^{\text{obj}}$, and $\text{sPr}(k)_*^{\text{loc}} := */\text{sPr}(k)^{\text{loc}}$.
- (2) ([22, 25]) We say a pointed simplicial presheaf $X \in \text{sPr}(k)_*$ is a pointed schematic homotopy type if the following conditions are satisfied.

1. X is a connected local object.
2. The homotopy sheaf $\pi_i(X, *)$ is represented by an affine group scheme over k for any $i \geq 1$ and it is unipotent for $i \geq 2$.

We say a pointed schematic homotopy type $X \in \text{sPr}(k)_*$ is a pointed affine stack if $\pi_1(X, *)$ is represented by a unipotent affine group scheme over k . We denote the full subcategory of $\text{sPr}(k)_*$ consisting of pointed schematic homotopy types by SHT_* . $\text{Ho}(\text{SHT}_*)$ denotes the corresponding full subcategory of $\text{Ho}(\text{sPr}(k)_*^{\text{obj}})$ (or $\text{Ho}(\text{sPr}(k)_*^{\text{loc}})$). We say an object $U \in \text{sPr}(k)$ is a schematic homotopy type if there exists a point $*$ $\in U(k)_0$ such that the pointed object $(U, *) \in \text{sPr}(k)_*$ is a pointed schematic homotopy type. We denote by SHT the full subcategory of $\text{sPr}(k)$ consisting of schematic homotopy types.

(3) ([22]) Let $K \in \text{sSet}$ (resp. sSet_*) be a connected simplicial set. We denote by \underline{K} be a constant simplicial presheaf given by $\underline{K}(R) = K$ for $R \in k - \text{Alg}$. The schematization of K is a morphism

$$\underline{K} \longrightarrow (K \otimes k)^{\text{sch}} \in \text{Ho}(\text{sPr}(k)^{\text{obj}})$$

(resp. $\text{Ho}(\text{sPr}(k)_*^{\text{obj}})$) which is initial in the category $\underline{K}/\text{Ho}(\text{SHT})$ (resp. $\underline{K}/\text{Ho}(\text{SHT}_*)$). Here, $\underline{K}/\text{Ho}(\text{SHT})$ is the full subcategory of the over category $\underline{K}/\text{Ho}(\text{sPr}(k))$ consisting of objects $\underline{K} \rightarrow X$ such that X is a schematic homotopy type. $\underline{K}/\text{Ho}(\text{SHT}_*)$ is similar.

The above definition of pointed schematic homotopy types is different from the original one but they are shown to be equivalent, see [25, Cor.3.16]. The existence of the schematization in the pointed category was proved in [22] (we will reprove this later, based on the above definition). The existence of the unpointed schematization follows from Cor.5.2.7.

We shall define an adjoint pair $(\text{TPL}, \langle - \rangle) : \text{sPr}(k) \rightarrow (\text{dgCat}^{\text{cl}})^{\text{op}}$ (or $\text{sPr}(k)_* \rightarrow (\text{dgCat}_*^{\text{cl}})^{\text{op}}$). This is an analogue of the adjoint pair $(\text{TPL}, \langle - \rangle) : \text{sSet} \rightarrow (\text{dgCat}^{\text{cl}})^{\text{op}}$ defined in subsection 3.4.

We first define a notion of local systems on a simplicial presheaf. In order to ensure some functoriality, we take care about the definition of the category of projective modules. The following definition of (1) can be found in [23, 1.3.7], for example. Let $R - \text{Mod}$ denote the category of R -modules and R -linear maps.

Definition 5.1.2. (1) For $R \in k - \text{Alg}$ we define a category $R - \text{Proj}$ as follows.

- An object is a pair (M_-, ϕ_-) consisting of a function M_- which assigns each R -algebra S a finitely generated projective S -module M_S and a function ϕ_- which assigns each morphism $f : S \rightarrow S' \in R - \text{Alg}$ an isomorphism $\phi_f : M_S \otimes_S S' \rightarrow M_{S'} \in S' - \text{Mod}$ such that for each sequence $S \xrightarrow{f} S' \xrightarrow{f'} S'' \in R - \text{Alg}$ the following diagram is commutative.

$$\begin{array}{ccc} (M_S \otimes_S S') \otimes_{S'} S'' & \xrightarrow{\phi_f \otimes S''} & M'_S \otimes_{S'} S'' \\ \downarrow & & \downarrow \phi_{f'} \\ M_S \otimes_S S'' & \xrightarrow{\phi_{f' \circ f}} & M''_S \end{array}$$

Here, the left vertical arrow is induced from the associativity isomorphism and unit isomorphism. Sometimes we omit ϕ .

- A morphism $F : (M, \phi) \rightarrow (N, \varphi)$ is a function assigning each R -algebra S a morphism $F_S : M_S \rightarrow N_S \in S - \text{Mod}$ such that for any $f : S \rightarrow S' \in R - \text{Alg}$ the following diagram is commutative

$$\begin{array}{ccc} M_S \otimes_S S' & \xrightarrow{F_S \otimes S'} & N_S \otimes_S S' \\ \downarrow \phi_f & & \downarrow \varphi_f \\ M_{S'} & \xrightarrow{F_{S'}} & N_{S'} \end{array}$$

We regard $R - \text{Proj}$ as a closed k -category in the obvious way. For example the internal hom $\mathfrak{Hom}(M, N)$ is given by $\mathfrak{Hom}(M, N)_S = \mathfrak{Hom}(M_S, N_S)$, the right hand side is the internal hom in $S - \text{Mod}$. For $f : R \rightarrow R' \in k - \text{Alg}$ we have a morphism $f_* : R - \text{Proj} \rightarrow R' - \text{Proj} \in R - \text{Cat}^{\text{cl}}$ defined by $f_*(M)'_S = M_{f^* S'}$, where $S' \in R' - \text{Alg}$. Note that $f'_* \circ f_* = (f' \circ f)_*$.

(2) Let X be an object of $\text{sPr}(k)$. Let $R - \text{Proj}^{\text{iso}}$ be the full subcategory of $R - \text{Proj}$ consisting of all objects and all isomorphisms.

A local system \mathcal{L} on X is a collection $\{\mathcal{L}_R\}_{R \in k - \text{Alg}}$ of functors $\mathcal{L}_R : [\Delta(X(R))]^{\text{op}} \rightarrow R - \text{Proj}^{\text{iso}}$ such that for each morphism $f : R \rightarrow R' \in k - \text{Alg}$ the following diagram is commutative.

$$\begin{array}{ccc} [\Delta(X(R))]^{\text{op}} & \xrightarrow{\mathcal{L}_R} & R - \text{Proj} \\ \downarrow (\Delta X_f)^{\text{op}} & & \downarrow f_* \\ [\Delta(X(R'))]^{\text{op}} & \xrightarrow{\mathcal{L}_{R'}} & R' - \text{Proj} \end{array}$$

A morphism $\alpha : \mathcal{L} \rightarrow \mathcal{L}'$ of local systems is a collection $\{\alpha_R\}$ of natural transformations $\alpha_R : \mathcal{L}_R \Rightarrow \mathcal{L}'_R : [\Delta(X(R))]^{\text{op}} \rightarrow R - \text{Proj}$ compatible with ring homomorphisms. We denote by $\text{Loc}(X)$ the category of local systems on X . We regard $\text{Loc}(X)$ as a closed k -category in the obvious way.

Let $X \in \text{sPr}(k)_*$ be a connected object such that $\pi_1(X, *)$ is represented by an affine group scheme G over k . It is easy to see $\text{Loc}(X)$ is equivalent to the category of finite dimensional representations of G .

Definition 5.1.3. Let X be a simplicial presheaf and \mathcal{L} be a local system on X . The de Rham complex of \mathcal{L} -valued polynomial forms $C_{\text{PL}}(X; \mathcal{L}) \in C^{\geq 0}(k)$ is defined as follows. For each $q \geq 0$, the degree q part is the submodule of

$$\prod_{R \in k - \text{Alg}} \prod_{p \geq 0} \prod_{\sigma \in X(R)_p} \nabla(p, q) \otimes_k \mathcal{L}_R(\sigma)_R$$

consisting of $\{\omega_\sigma\}_{R \in k - \text{Alg}, \sigma \in \Delta X(R)}$'s such that

1. for each algebra $R \in k - \text{Alg}$ and morphism $a : \tau \rightarrow \sigma \in \Delta X(R)$, $a^* \omega_\sigma = \omega_\tau$, and
2. for each map $f : R \rightarrow R' \in k - \text{Alg}$ and simplex $\sigma \in X(R)$, $(\text{id} \otimes_k f_*)(\omega_\sigma) = \omega_{X_f \sigma}$.

Here, a^* denotes the tensor of $a^* : \nabla(|\sigma|, q) \rightarrow \nabla(|\tau|, q)$ and $\mathcal{L}_R(a) : \mathcal{L}_R(\sigma) \rightarrow \mathcal{L}_R(\tau)$ ($|\cdot|$ denote the dimension of a simplex), and $(\text{id} \otimes_k f_*)(\omega_\sigma)$ denotes the image of ω_σ by the tensor of the identity of $\nabla(|\sigma|, q)$ and the following composition.

$$\mathcal{L}_R(\sigma)_R \rightarrow \mathcal{L}_R(\sigma)_R \otimes_R R' \xrightarrow{\phi_f} \mathcal{L}_R(\sigma)_{R'} = \mathcal{L}_{R'}(X_f \sigma).$$

For $q \leq -1$, we set $C_{\text{PL}}^q(X, \mathcal{L}) = 0$. The differential $d : C_{\text{PL}}^q(X, \mathcal{L}) \rightarrow C_{\text{PL}}^{q+1}(X, \mathcal{L})$ is defined from that of $\nabla(*, q)$.

We define a functor

$$T_{\text{PL}} : \text{sPr}(k) \longrightarrow (\text{dgCat}^{\text{cl}})^{\text{op}}$$

by setting

$$\text{Ob}(T_{\text{PL}}(X)) = \text{Ob}(\text{Loc}(X)), \quad \text{Hom}_{T_{\text{PL}}(X)}(\mathcal{L}, \mathcal{L}') = C_{\text{PL}}(X; \mathfrak{Hom}(\mathcal{L}, \mathcal{L}'))$$

for $X \in \text{sPr}(k)$. The composition and the closed tensor structure are defined in the obvious way.

A right adjoint

$$\langle - \rangle : (\text{dgCat}^{\text{cl}})^{\text{op}} \longrightarrow \text{sPr}(k)$$

of T_{PL} is defined by

$$\langle C \rangle(R)_n := \text{Hom}_{\text{dgCat}^{\text{cl}}}(C, T_{\text{PL}}(h_R \times \Delta^n))$$

for $C \in \text{dgCat}^{\text{cl}}$, $n \geq 0$ and $R \in k - \text{Alg}$. Here h_R denotes the Yoneda embedding of $\text{Spec}R$, and $h_R \times \Delta^n$ is a simplicial presheaf given by $h_R \times \Delta^n(S) := h_R(S) \times \Delta^n \in \text{sSet}$. For a pointed simplicial presheaf $X \in \text{sPr}(k)_*$, we define a closed dgc with a fiber functor $T_{\text{PL}}(X) \in \text{dgCat}_*^{\text{cl}}$ by the following pullback square:

$$\begin{array}{ccc} T_{\text{PL}}(X) & \longrightarrow & T_{\text{PL}}(X_u) \\ \downarrow & & \downarrow T_{\text{PL}}(pt) \\ \text{Vect} & \longrightarrow & T_{\text{PL}}(*), \end{array}$$

where $X_u \in \text{sPr}(k)$ denotes the underlying unpointed simplicial presheaf. On the other hand, for an object $C \in \text{dgCat}_*^{\text{cl}}$, $\langle C \rangle$ is naturally pointed as $\langle \text{Vect} \rangle \cong *$ so we have an adjoint pair $(T_{\text{PL}}, \langle - \rangle) : \text{sPr}(k)_* \rightarrow (\text{dgCat}_*^{\text{cl}})^{\text{op}}$. For connected object $X \in \text{sPr}(k)_*$, $T_{\text{PL}}(X)$ is equivalent to $T_{\text{PL}}(X_u)$ as closed dgc's.

Lemma 5.1.4. *The adjoint pairs $(T_{PL}, \langle - \rangle) : sPr(k)^{obj} \rightarrow (dgCat^{cl})^{op}$ and $(T_{PL}, \langle - \rangle) : sPr(k)_*^{obj} \rightarrow (dgCat_*^{cl})^{op}$ are Quillen pairs.*

Proof. When we use Prop.A.1.2, the proof is similar to that of Lem.3.4.3. Note that for a local system \mathcal{L} on $h_R \times \Delta^n$, $C_{PL}(h_R \times \Delta^n, \mathcal{L}') \cong C_{PL}(\Delta^n; \mathcal{L}_R|_{id_R \times \Delta^n})$. \square

5.2 Equivalence between schematic homotopy types and Tannakian dg-categories

In this subsection, we prove the following theorem. Note that for connected $K \in sSet_*$ $\mathbb{L}T_{PL}(\underline{K})$ is naturally (quasi-)equivalent to $T_{PL}(K)$ which is defined in 3.4.

Theorem 5.2.1 (cf. Cor.3.57 of [27]). (1) *The derived adjunction $(\mathbb{L}T_{PL}, \mathbb{R}\langle - \rangle) : \mathbf{Ho}(sPr(k)_*^{obj}) \rightarrow \mathbf{Ho}(dgCat_*^{cl})$ induces an equivalence between full subcategories:*

$$\mathbb{L}T_{PL} : \mathbf{Ho}(SHT_*) \rightleftarrows \mathbf{Ho}(\mathbf{Tan}_*)^{op} : \mathbb{R}\langle - \rangle.$$

(2) *For connected pointed simplicial set K , the unit of the adjunction*

$$\underline{K} \longrightarrow \mathbb{R}\langle T_{PL}(K) \rangle \in \mathbf{Ho}(sPr(k)_*^{obj})$$

is the schematization over k .

We prove this using the following theorem of [22].

Theorem 5.2.2 (Thm.0.0.2 and Cor.2.2.3 of [22]). *Let $cAlg/k$ be the category of commutative and unital cosimplicial k -algebras with augmentations. Let $\mathcal{O} : sPr(k)_*^{loc} \rightarrow (cAlg/k)^{op}$ be the functor such that for a simplicial presheaf $X \in sPr(k)_*^{loc}$, $\mathcal{O}(X)^n = \text{Hom}(X_n, \mathcal{O}_0)$, where \mathcal{O}_0 is the tautological presheaf $k - Alg \ni R \mapsto R \in Set$ and Hom is the set of morphisms between presheaves. Then \mathcal{O} is a left Quillen functor and its derived functor $\mathbb{L}\mathcal{O} : \mathbf{Ho}(sPr(k)_*^{loc}) \rightarrow \mathbf{Ho}(cAlg/k)^{op}$ induces an equivalence between the full subcategory of affine stacks and one of connected (i.e. $H^0 \cong k$) cosimplicial algebras.*

This theorem can be formulated using dg-algebras instead of cosimplicial algebras as follows. Let $A_{PL} : sPr(k)_*^{loc} \rightarrow (dgAlg/k)^{op}$ be the functor defined by $A_{PL}(X) = C_{PL}(X, \mathbf{1})$. This is a left Quillen functor whose right adjoint $| - |$ is given by $|A|(R)_n := \text{Hom}_{dgAlg}(A, A_{PL}(h_R \times \Delta^n))$. It is well-known that the Thom-Sullivan cochain functor $Th : cAlg/k \rightarrow dgAlg/k$ induces equivalence of homotopy categories, and it is easy to see the following diagram commutative up to natural quasi-isomorphism.

$$\begin{array}{ccc} sPr(k)_*^{loc} & \xrightarrow{\mathcal{O}} & (cAlg/k)^{op} \\ & \searrow A_{PL} & \downarrow Th \\ & & (dgAlg/k)^{op}. \end{array}$$

So we may rephrase Thm.5.2.2 that the functor $\mathbb{L}A_{PL} : \mathbf{Ho}(sPr(k)_*^{loc}) \rightarrow \mathbf{Ho}(dgAlg/k)^{op}$ induces an equivalence between affine stacks and 0-connected dga's.

We shall state some fundamental results.

Definition 5.2.3. *For a pointed schematic homotopy type X , we put $A_{red}(X) := A_{red}(\mathbb{L}T_{PL}(X))$*

Proposition 5.2.4. (1) *Let $X \in sPr(k)_*$. $\mathbb{L}T_{PL}X$ is complete. If X is connected, $\mathbb{L}T_{PL}X$ is a Tannakian dg-category with a fiber functor. $\mathbb{L}T_{PL}X$ and $T A_{red}(X)$ are equivalent.*

(2) *For $T \in \mathbf{Tan}_*$, let T^{ss} denote the full sub dgc of T consisting of semisimple objects of $Z^0 T$. The inclusion*

$T^{\text{ss}} \longrightarrow T$ induces an isomorphism $\mathbb{R}\langle T \rangle \longrightarrow \mathbb{R}\langle T^{\text{ss}} \rangle \in \mathbf{Ho}(\mathbf{sPr}(k)_*^{\text{obj}})$.

(3) For any object $T \in \mathbf{Tan}_*$, $\mathbb{R}\langle T \rangle \in \mathbf{Ho}(\mathbf{sPr}(k)_*^{\text{obj}})$ is connected.

(4) Let $X, Y \in \mathbf{sPr}(k)_*$ be connected objects which is cofibrant in $\mathbf{sPr}(k)_*^{\text{obj}}$. For a local equivalence $f : X \rightarrow Y$, the corresponding morphism $f^* : \mathbf{TPL}Y \rightarrow \mathbf{TPL}X$ is a quasi-equivalence.

(5) Let $F \in \mathbf{sPr}(k)_*$ be a pointed affine stack. The unit of the adjoint

$$F \longrightarrow \mathbb{R}\langle \mathbf{LTPL}(F) \rangle \in \mathbf{Ho}(\mathbf{sPr}(k)_*^{\text{obj}})$$

is an isomorphism.

Proof. The proof of (1) is similar to that of Thm.3.4.5. For (2), by (1) and Lem.2.2.4, for any $L \in \mathbf{sPr}(k)$,

$$[L, \mathbb{R}\langle T^{\text{ss}} \rangle]_{\mathbf{sPr}(k)} \cong [T^{\text{ss}}, \mathbf{LTPL}(L)]_{\mathbf{dgCat}^{\text{cl}}} \cong [T, \mathbf{LTPL}(L)]_{\mathbf{dgCat}^{\text{cl}}} \cong [L, \mathbb{R}\langle T \rangle]_{\mathbf{sPr}(k)}.$$

(3) follows from [11, Thm.3.2]. (4) follows from Prop.A.1.2 as invariance of $C_{\text{spl}}(-; \mathcal{L})$ under local equivalence between connected objects are well-known (see e.g.[22]). (5) follows from the reformulation of Thm.5.2.2 and (2) as $\mathbb{R}\langle \mathbf{LTPL}F \rangle \cong \mathbb{R}\langle \mathbf{T}^{\text{ss}} \mathbf{A}_{\mathbf{PL}}(QF) \rangle \cong \mathbb{R}|\mathbf{A}_{\mathbf{PL}}(QF)| \cong F$. \square

For an affine group scheme G , define a simplicial presheaf $K(G, 1) \in \mathbf{sPr}(k)_*$ by $K(G, 1)(R) := K(G(R), 1)$.

Lemma 5.2.5. (1) Let G be a reductive affine group scheme. Let $K \in \mathbf{sPr}(k)_*$ be a fibrant model of $K(G, 1)$ in $\mathbf{sPr}(k)_*^{\text{loc}}$. The unit of the adjoint

$$K \longrightarrow \mathbb{R}\langle \mathbf{LTPL}(K) \rangle \in \mathbf{Ho}(\mathbf{sPr}(k)_*^{\text{obj}})$$

is a local equivalence. (Note that local equivalences are stable under objectwise equivalences so we apply the notion to the morphism of $\mathbf{Ho}(\mathbf{sPr}(k)_*^{\text{obj}})$)

(2) Let X be a pointed connected simplicial presheaf such that $\pi_0(X) \cong *$ and $\pi_1(X)$ is represented by an affine group scheme G . Suppose X is fibrant in $\mathbf{sPr}(k)_*^{\text{loc}}$. Let $K \in \mathbf{sPr}(\mathbb{Q})_*$ be a fibrant model of $K(G^{\text{red}}, 1)$ in $\mathbf{sPr}(k)_*^{\text{loc}}$ and $p : X \longrightarrow K \in \mathbf{sPr}(k)_*$ be a fibration such that $\pi_1(p) : \pi_1(X) \longrightarrow \pi_1(K)$ is isomorphic to the canonical map $G \longrightarrow G^{\text{red}}$. Let $F \in \mathbf{sPr}(k)_*$ be the fiber of p at the base point. Then the sequence

$$\mathbb{R}\langle \mathbf{LTPL}(F) \rangle \longrightarrow \mathbb{R}\langle \mathbf{LTPL}(X) \rangle \longrightarrow \mathbb{R}\langle \mathbf{LTPL}(K) \rangle$$

is a homotopy fiber sequence in $\mathbf{sPr}(k)_*^{\text{obj}}$.

Proof. Let QF be a cofibrant replacement of F . We first show $\mathbf{A}_{\text{red}}(QF)$ ($\cong C_{\mathbf{PL}}(QF; \mathbf{1})$) is quasi-isomorphic to $\mathbf{A}_{\text{red}}(X)$ with trivial group action as a dga. As an explicit model of F we use \tilde{X} defined as follows. We denote by $\mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}})$ the presheaf of groupoid of fiber functors of $\mathbf{Loc}(X)^{\text{ss}}$, the category of semi-simple local systems on X . Precisely, for an algebra $R \in k - \mathbf{Alg}$, An object of $\mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}})(R)$ is a morphism $\omega : \mathbf{Loc}(X)^{\text{ss}} \rightarrow R - \mathbf{Proj} \in \mathbf{Cat}^{\text{cl}}$. and a morphism is a natural transformation preserving tensors. By the definition of $R - \mathbf{Proj}$ the correspondence $R \mapsto \mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}})(R)$ is functorial. Let $N(\mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}}))$ be the corresponding simplicial presheaf obtained by taking nerve in the objectwise manner. This is a fibrant model of $K(G^{\text{red}}, 1)$ in $\mathbf{sPr}(k)_*^{\text{loc}}$ ($G = \pi_1(X)$). We have a natural morphism

$$p' : X \longrightarrow N(\mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}}))$$

such that $\pi_1(p') : \pi_1(X) \longrightarrow \pi_1(N(\mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}}))$ is isomorphic to $G \longrightarrow G^{\text{red}}$. We set

$$\tilde{X}(R)_n = \{(\sigma, l) | \sigma \in X(R)_n, l \in \mathbf{Fib}(\mathbf{Loc}(X)^{\text{ss}})(\omega_{\sigma(0)}, \omega_*)\}$$

where $\omega_x : \text{Loc}(X)^{\text{ss}} \rightarrow R - \text{Proj}$ denotes the R -fiber functor comes from $x \in X(R)_0$. Note that \tilde{X} has a G^{red} -action given by $g \cdot (\sigma, l) = (\sigma, g \circ l)$. We shall show $\text{A}_{\text{red}}(\tilde{X})$ is isomorphic to $\text{A}_{\text{red}}(X)$. $\mathcal{O}(G^{\text{red}})$ is considered as a local system on X by

$$X(R)_n \ni \sigma \longmapsto \mathcal{O}(\underline{\text{Hom}}_{Fib(\text{Loc}(X)^{\text{ss}})}(\omega_{\sigma(0)}, \omega_*)).$$

Here, $\underline{\text{Hom}}_{Fib(\text{Loc}(X)^{\text{ss}})}(-, -)$ denotes the sheaf of natural transformations which is represented by an affine scheme over R . Note that if X is cofibrant, \tilde{X} is also cofibrant.

A map $\phi : \text{A}_{\text{red}}(X) \rightarrow \text{A}_{\text{red}}(\tilde{X})$ is defined as follows. An element of $\text{A}_{\text{red}}(X)$ is presented as $\sum_i \alpha_i \otimes f_i$, $\alpha_i \in \nabla(n, *)$, $f_i \in \mathcal{O}(\underline{\text{Hom}}_{Fib(\text{Loc}(X)^{\text{ss}})}(\omega_*, \omega_{\sigma(0)}))$ locally on $\sigma \in X(R)_n$. We define an element $\phi(\sum_i \alpha_i \otimes f_i)$ to be $\sum_i \alpha_i \otimes l(f_i) \in \nabla(n, *) \otimes R$ on (σ, l) . Here, l is considered as a map $\mathcal{O}(\underline{\text{Hom}}_{Fib(\text{Loc}(X)^{\text{ss}})}(\omega_{\sigma(0)}, \omega_*)) \rightarrow R$.

A map $\varphi : \text{A}_{\text{red}}(\tilde{X}) \rightarrow \text{A}_{\text{red}}(X)$ is defined as follows. $\text{A}_{\text{red}}(\tilde{X})$ has an G^{red} -action induced from the action on \tilde{X} . The corresponding comoduled map induce an isomorphism $\rho : \text{A}_{\text{red}}(\tilde{X}) \cong (\mathcal{O}(G^{\text{red}}) \otimes \text{A}_{\text{red}}(\tilde{X}))^{G^{\text{red}}}$. Via this isomorphism, an element of $\text{A}_{\text{red}}(\tilde{X})$ are presented as $\sum_j v_j \otimes a_j$, $v_j \in \mathcal{O}(G^{\text{red}})$, $a_j \in \text{A}_{\text{red}}(\tilde{X})$. We define an element $\varphi(\sum_j v_j \otimes a_j)$ to be $\sum_j a_j(\sigma, l) \otimes_R [(l \circ \iota \otimes \text{id}) \circ \mu^*(v_j)]$ locally on σ ($l \in Fib(\text{Loc}(X)^{\text{ss}})(\omega_{\sigma(0)}, \omega_*)$ is arbitrarily fixed.) Here,

$$\mu^* : \mathcal{O}(G^{\text{red}}) \cong \mathcal{O}(\underline{\text{Hom}}(\omega_*, \omega_*)) \rightarrow \mathcal{O}(\underline{\text{Hom}}(\omega_*, \omega_{\sigma(0)})) \otimes_R \mathcal{O}(\underline{\text{Hom}}(\omega_{\sigma(0)}, \omega_*))$$

is the cocomposition and

$$\iota : \mathcal{O}(\underline{\text{Hom}}(\omega_*, \omega_{\sigma(0)})) \rightarrow \mathcal{O}(\underline{\text{Hom}}(\omega_{\sigma(0)}, \omega_*))$$

is the coinverse map. ϕ and φ are morphisms of G^{red} -equivariant dga and inverse to each other. Thus, $\text{A}_{\text{red}}(\tilde{X}) \cong \text{A}_{\text{red}}(X)$. Consider the case that G is reductive and X is a fibrant model of $K(G, 1)$ in $s\text{Pr}(k)_*^{\text{loc}}$. Clearly F is contractible so $\text{A}_{\text{red}}(X) \simeq (G, k)$. and $\text{LT}_{\text{PL}}(K) \cong \text{T}^{\text{ss}}(G^{\text{red}}, k)$ ($\cong \text{Rep}(G^{\text{red}})$). Then,

$$\pi_n^{pre}(\mathbb{R}\langle \text{T}_{\text{PL}}(K) \rangle)(R) \cong [h_R \wedge S^n, \mathbb{R}\langle \text{T}_{\text{PL}}(K) \rangle]_{s\text{Pr}(k)_*^{\text{obj}}} \cong [\text{T}^{\text{ss}}(G, k), \text{T}_{\text{PL}}(h_R \wedge S^n)]_{\text{dgCat}_*^{\text{cl}}}$$

Here, $h_R \wedge S^n \in s\text{Pr}(k)_*$ be an object defined by the following pushout diagram

$$\begin{array}{ccc} h_R & \longrightarrow & * \\ \downarrow \text{id} \times v & & \downarrow \\ h_R \times S^n & \longrightarrow & h_R \wedge S^n, \end{array}$$

where v is the base point of S^n . If $n \geq 2$, this is clearly $*$. If $n = 1$, The last term is bijective to the set

$$\text{Hom}_{\text{Cat}^{\text{cl}}/R - \text{Proj}}(\text{T}^{\text{ss}}(G, k)^c, \text{Rep}_R(\mathbb{Z}))/\sim$$

Here $\text{Rep}_R(\mathbb{Z})$ is the closed k -category defined as follows.

1. An object of $\text{Rep}_R(\mathbb{Z})$ is a pair (V, r) of $V \in \text{Vect}$ and R -linear representation $r : \mathbb{Z} \rightarrow \text{GL}_R(V \otimes_k R)$ of \mathbb{Z} , and
2. a morphism $f : (V_0, r_0) \rightarrow (V_1, r_1)$ is a morphism $f : V_0 \rightarrow V_1$ of Vect such that $f \otimes_k R$ is compatible with the actions.

The augmentation $\omega_{\text{Rep}_R(\mathbb{Z})} : \text{Rep}_R(\mathbb{Z}) \rightarrow R - \text{Proj}$ is given by the forgetful functor. \sim is an equivalence relation such that $F_1 \sim F_2$ if and only if there is a natural isomorphism $i : F_1 \Rightarrow F_2$ such that i preserves tensors and $(\omega_{\text{Rep}_R(\mathbb{Z})})_* i$ is the identity. This set is bijective to $\text{Aut}^{\otimes}(\omega_{\text{T}^{\text{ss}}(G, k)^c} : \text{T}^{\text{ss}}(G, k)^c \rightarrow R - \text{Proj})$ in

the way that an action of $1 \in \mathbb{Z}$ corresponds a natural automorphism. Thus we have $\pi_1^{pre}(\mathbb{R}\langle \mathrm{TPL}(K) \rangle) \cong \underline{\mathrm{Aut}}^\otimes(\omega) \cong G$ and obtain (1) (see Appendix A.2).

By (1), Thm.3.3.4 and Lem.3.3.5, there is a commutative diagram in $\mathrm{dgCat}_*^{\mathrm{cl}}$:

$$\begin{array}{ccccc} \mathrm{T}^{\mathrm{ss}}(G, k) & \longrightarrow & \mathrm{T}^{\mathrm{ss}}(G, \mathrm{A}_{\mathrm{red}} X) & \longrightarrow & \mathrm{T}^{\mathrm{ss}}(e, \mathrm{A}_{\mathrm{red}} X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TPL}(K)^{\mathrm{ss}} & \longrightarrow & \mathrm{TPL}(X)^{\mathrm{ss}} & \longrightarrow & \mathrm{TPL}(QF)^{\mathrm{ss}} \end{array}$$

such that the left and middle vertical arrows are quasi-equivalences. By the above assertion, the right vertical arrow is also a quasi-equivalence so the bottom horizontal sequence is a homotopy cofiber sequence and this imply the claim of (2). \square

Proof of Thm.5.2.1. Prop.5.2.4 (5), Lem.5.2.5 and the long exact sequence of homotopy sheaves show the unit

$$u_X : X \longrightarrow \mathbb{R}\langle \mathrm{LTPL}(X) \rangle$$

is a local equivalence for $X \in \mathrm{SHT}_*$. Consider the commutative diagram in $\mathrm{Ho}(\mathrm{sPr}(k)_*^{\mathrm{obj}})$:

$$\begin{array}{ccc} \mathbb{R}\langle \mathrm{LTPL}(X) \rangle & \xrightarrow{u} & \mathbb{R}\langle \mathrm{LTPL}(\mathbb{R}\langle \mathrm{LTPL}(X) \rangle) \rangle \\ \downarrow i & & \downarrow \mathbb{R}\langle \mathrm{LTPL}(i) \rangle \\ R_{loc}\mathbb{R}\langle \mathrm{LTPL}(X) \rangle & \xrightarrow{u'} & \mathbb{R}\langle \mathrm{LTPL}(R_{loc}\mathbb{R}\langle \mathrm{LTPL}(X) \rangle) \rangle, \end{array}$$

where $u = u_{\mathbb{R}\langle \mathrm{LTPL}(X) \rangle}$, $u' = u_{R_{loc}\mathbb{R}\langle \mathrm{LTPL}(X) \rangle}$, and i is a fibrant replacement in $\mathrm{sPr}(k)_*^{\mathrm{loc}}$. By Prop.5.2.4,(3), $\mathbb{R}\langle \mathrm{LTPL}(i) \rangle \circ u$ is an isomorphism so $u' \circ i$ is and we may regard $\mathbb{R}\langle \mathrm{LTPL}(X) \rangle$ as a retract of $R_{loc}\mathbb{R}\langle \mathrm{LTPL}(X) \rangle$ in $\mathrm{Ho}(\mathrm{sPr}(k)_*^{\mathrm{obj}})$. By a characterization of local objects, this imply $\mathbb{R}\langle \mathrm{LTPL}(X) \rangle$ is also a local object. As a local equivalence between local objects is an objectwise equivalence, we see u_X is an isomorphism. Thus $\mathrm{LTPL} : \mathrm{Ho}(\mathrm{SHT}_*) \longrightarrow \mathrm{Ho}(\mathrm{Tan})$ is fully faithful. By Lem.3.3.5 and 5.2.5, we see a morphism $f : T \rightarrow T' \in \mathrm{Tan}$ is a quasi-equivalence if and only if $\mathbb{R}\langle f \rangle : \mathbb{R}\langle T' \rangle \rightarrow \mathbb{R}\langle T \rangle$ is a weak equivalence so LTPL is essentially surjective. \square

The following is a corollary of Thm.5.2.1.

Corollary 5.2.6 (cf. Rem.4.43 of [27]). *Let $X \in \mathrm{SHT}_*$ be a schematic homotopy type and \mathcal{M} be a minimal model of $\mathrm{A}_{\mathrm{red}}(X)$. Let V^i denote the i -th indecomposable module of \mathcal{M} . For $i \geq 2$, there exists an isomorphism of affine group schemes : $\pi_i(X) \cong (V^i)^\vee$. For $i = 1$, there is an isomorphism of affine schemes (without group structure) $R_u(\pi_1(X)) \cong (V^1)^\vee$. Here, we regard $(V^i)^\vee$ as an affine group scheme whose coordinate ring is the polynomial ring on V^i .*

Proof. Let $i \geq 2$. Then, by Thm.5.2.1

$$\pi_i^{pre}(X)(R) \cong \pi_i^{pre}(\mathbb{R}\langle \mathrm{LTPL}(X) \rangle)(R) \cong [h_R \wedge S^i, \mathbb{R}\langle \mathrm{LTPL}(X) \rangle]_{\mathrm{sPr}(k)_*^{\mathrm{obj}}} \cong [\mathrm{LTPL}(X), \mathrm{LTPL}(h_R \wedge S^i)]_{\mathrm{dgCat}_*^{\mathrm{cl}}},$$

see the proof of Lem.5.2.5 for the notation. If we define $A_i^R \in \mathrm{dgAlg}/k$ by $(A_i^R)^0 = k, (A_i^R)^i = R$, and $(A_i^R)^j = 0$ for $j \neq 0, i$, we easily see $\mathrm{T}(e, A_i^R) \simeq \mathrm{LTPL}(h_R \wedge S^i)$ so by Lem.3.3.7, $\pi_i^{pre}(X)(R) \cong [\mathcal{M}, A_i^R]_{\mathrm{dgAlg}/k} \cong \mathrm{Hom}_{k-\mathrm{Mod}}(V^i, R)$ and claim for $i \geq 2$ follows. The case of $i = 1$ follows from a similar argument using the homotopy fiber F of the map $X \rightarrow K(\pi_1(X)^{\mathrm{red}}, 1)$. Note that $\pi_1(F) \cong R_u(\pi_1(X))$ and $\mathrm{A}_{\mathrm{red}}(F) \cong \mathrm{A}_{\mathrm{red}}(X)$ (see the proof of Lem.5.2.5). \square

We obtain the unpointed version of Thm.5.2.1

Corollary 5.2.7. (1) *The derived adjunction $(\mathbb{L}T_{PL}, \mathbb{R}\langle - \rangle) : \text{Ho}(\text{sPr}(k)^{\text{obj}}) \longrightarrow \text{Ho}(\text{dgCat}^{\text{cl}})^{\text{op}}$ induces an equivalence between subcategories:*

$$\mathbb{L}T_{PL} : \text{Ho}(\text{SHT}) \rightleftarrows \text{Ho}(\text{Tan})^{\text{op}} : \mathbb{R}\langle - \rangle.$$

(2) *Any connected simplicial set $K \in \text{sSet}$ admits the schematization $\underline{K} \rightarrow (K \otimes k)^{\text{sch}}$ in the unpointed category. It is realized as the unit of the adjunction $\underline{K} \longrightarrow \mathbb{R}\langle T_{PL}(K) \rangle \in \text{Ho}(\text{sPr}(k)^{\text{obj}})$.*

Proof. This is clear from Thm.5.2.1. \square

A Appendix

A.1 Variants of simplicial de Rham theorem

In this subsection, we state some variants of de Rham theorem for simplicial sets, which is used in sections 4 and 5. Their proofs are much similar to the proof in [6, section 2, section 3], so we only indicate how to modify.

A.1.1 Twisted de Rham theorem

Let K be a simplicial set and \mathcal{L} be a local system on K . We denote by $C_{\text{spl}}(K; \mathcal{L})$ the normalized cochain complex of K with \mathcal{L} coefficients. An element of the degree q -part $C_{\text{spl}}^q(K; \mathcal{L})$ is a function u which assigns each q -simplex $\sigma \in K_q$ an element $u(\sigma) \in \mathcal{L}(\sigma)$ such that for any degenerate simplex σ , $u(\sigma) = 0$.

We have the Stokes map as usual:

$$\rho_{K, \mathcal{L}} : C_{\text{PL}}(K; \mathcal{L}) \longrightarrow C_{\text{spl}}(K; \mathcal{L}) \in C^{\geq 0}(k).$$

The twisted de Rham theorem is the following

Proposition A.1.1. *For a simplicial set K and a local system \mathcal{L} on K , $\rho_{K, \mathcal{L}}$ is a quasi-isomorphism.*

Proof. We have two functors:

$$C_{\text{PL}}(-; \mathcal{L}), C_{\text{spl}}(-; \mathcal{L}) : \text{sSet}/K \longrightarrow C^{\geq 0}(k)$$

defined by

$$(\phi : L \rightarrow K) \longmapsto C_{\text{PL}}(L; \phi^* \mathcal{L}), C_{\text{spl}}(L; \phi^* \mathcal{L}).$$

respectively. We take $\{\Delta^n \rightarrow X | n \geq 0\}$ as models instead of $\{\Delta^n\}$ and apply the argument in [6, section 2, section 3]. It is clear $C_{\text{PL}}(-; \mathcal{L})$ and $C_{\text{spl}}(-; \mathcal{L})$ is corepresentable and acyclic on models (in a suitably modified sense). \square

A.1.2 Twisted de Rham theorem for simplicial presheaves

We use notations defined in subsection 5.1. Let $X \in \text{sPr}(k)$ be a simplicial presheaf and \mathcal{L} be a local system on X . We define a non-negatively graded cochain complex $C_{\text{spl}}(X; \mathcal{L})$ as follows. An element of $C_{\text{spl}}^q(X; \mathcal{L})$ is a collection $\{u_R\}_{R \in k-\text{Alg}}$ such that $u_R \in C_{\text{spl}}^q(X(R); \mathcal{L}_R)$ and for $f : R \rightarrow R' \in k-\text{Alg}$ and $\sigma \in X(R)$, $f_*(u_R(\sigma)) = u_{R'}(X_f(\sigma))$. the differential is defined in the component-wise manner.

We have a Stokes map

$$\rho_{X, \mathcal{L}} : C_{\text{PL}}(X; \mathcal{L}) \longrightarrow C_{\text{spl}}(X; \mathcal{L}) \in C^{\geq 0}(k).$$

Let $I = \{h_R \times \partial\Delta^n \rightarrow h_R \times \Delta^n | R \in k-\text{Alg}, n \geq 0\}$, where h_R denotes the Yoneda embedding of $\text{Spec}R$.

Proposition A.1.2. *For an I-cell object $X \in \text{sPr}(k)$ and a local system \mathcal{L} on X , the Stokes map $\rho_{X,\mathcal{L}}$ is a quasi-isomorphism.*

Proof. The proof is similar to that of Prop.A.1.1. In this case, we take $\{h_R \times \Delta^n \rightarrow X\}$ as models. Note that for a local system \mathcal{L}' on $h_R \times \Delta^n$, $C_{\text{spl}}(h_R \times \Delta^n, \mathcal{L}') \cong C_{\text{spl}}(\Delta^n; \mathcal{L}_R|_{\text{id}_R \times \Delta^n})$ so $C_{\text{spl}}(-; \mathcal{L})$ is acyclic on models. \square

A.1.3 de Rham theorem for cubical sets

Let Set^\square be the category of cubical sets. An object of Set^\square consists of a collection of sets $\{K_n\}_{n \geq 0}$, face maps $\partial_i^\epsilon : K_n \rightarrow K_{n-1}$ ($0 \leq i \leq n$, $\epsilon = 0, 1$), and degeneracy maps $s_i : K_n \rightarrow K_{n+1}$ ($0 \leq i \leq n$) which satisfy the standard cubical identities. We denote by $\square^n \in \text{Set}^\square$ the standard n -cube. We regard Set^\square as a model category with the model structure given in [20, 21], where trivial fibrations are precisely those which have right lifting property with respect to the maps $\partial \square^n \rightarrow \square^n \in \text{Set}^\square$ ($n \geq 0$, of course, $\partial \square^n$ is the boundary of \square^n).

We denote by $\square(n, *)$ the dg-algebra of k -polynomial forms on \square^n . This is the commutative graded algebra over k freely generated by t_1, \dots, t_n and dt_1, \dots, dt_n with $\deg t_i = 0$, $\deg dt_i = 1$. (It is isomorphic to $\nabla(n, *)$.) For each $q \geq 0$, We regard $\square(*, q)$ as a cubical set (or cubical abelian group). Face maps and degeneracy maps are defined by the pullback of corresponding maps between the standard cubes. For example, $\partial_i^\epsilon : \square(n, q) \rightarrow \square(n-1, q)$ is defined as follows.

$$\partial_i^\epsilon(t_j) = \begin{cases} t_j & j < i \\ \epsilon & j = i \\ t_{j-1} & j > i \end{cases}$$

We need the following.

Lemma A.1.3. *For each $q \geq 0$, the unique morphism $\square(*, q) \rightarrow * \in \text{Set}^\square$ is a trivial fibration.*

Proof. We imitate the proof of fibrancy of simplicial abelian groups (see for example, [15, Lem.3.4]). We shall show the morphism of the claim has right lifting property with respect to the maps $\partial \square^n \rightarrow \square^n$ ($n \geq 0$). Suppose $2n$ elements $x_i^\epsilon \in \square(n-1, q)$, $i = 1, \dots, n$, $\epsilon = 0, 1$ such that $\partial_i^{\epsilon_1} x_j^{\epsilon_2} = \partial_{j-1} x_i^{\epsilon_1}$ for $i < j$, are given. (This is equivalent to giving a map $\partial \square^n \rightarrow \square(*, q) \in \text{Set}^\square$.) We use induction. Let $1 < l \leq n$. Suppose we have $y \in \square(n, q)$ such that for $l \leq \forall i \leq n$, $\forall \epsilon$, $\partial_i^\epsilon y = x_i^\epsilon$. (For $l = n$, put $y = (1 - t_n)s_n(x_n^0) + t_n s_n(x_n^1)$.)

Consider an element $z^\epsilon := x_{l-1}^\epsilon - \partial_{l-1}^\epsilon y$. Cubical identities imply $\forall i \geq l-1$, $\forall \epsilon'$, $\partial_i^{\epsilon'} z^\epsilon = 0$. Set $y' := (1 - t_{l-1})s_{l-1}z^0 + t_{l-1}s_{l-1}z^1 + y$. Again cubical identities imply $\partial_i^\epsilon y' = x_i^\epsilon$ for $l-1 \leq \forall i \leq n$, $\forall \epsilon$. Thus we can construct a lifting inductively. \square

Let $K \in \text{Set}^\square$. The notion of a local system on K is defined similarly to the case of simplicial sets. For a local system \mathcal{L} on K , we define the de Rham complex $C_{\text{PL}}(K; \mathcal{L}) \in C^{\geq 0}(k)$ using $\square(*, *)$. Let $C_\square(K; \mathcal{L})$ be the normalized cochain complex of K with \mathcal{L} -coefficients. Explicitly, an element of $C_\square^q(K; \mathcal{L})$ is a function u which assigns each q -cube $\sigma \in K_q$ an element $u(\sigma) \in \mathcal{L}(\sigma)$ such that $u(\sigma) = 0$ for a degenerate cube σ . The differential $d : C_\square^q(K; \mathcal{L}) \rightarrow C_\square^{q+1}(K; \mathcal{L})$ is as usual, given by

$$du = \sum_{i=1}^q (-1)^i [(\partial_i^0)^* u - (\partial_i^1)^* u].$$

Here, $(\partial_i^\epsilon)^* u = \mathcal{L}(\partial_i^\epsilon)^{-1} \circ u \circ \partial_i^\epsilon$. We have a Stokes map

$$\rho_{K, \mathcal{L}} : C_{\text{PL}}(K; \mathcal{L}) \longrightarrow C_\square(K; \mathcal{L}) \in C^{\geq 0}(k).$$

Proposition A.1.4. *For a cubical set K and a local system \mathcal{L} on K , $\rho_{K,\mathcal{L}}$ is a quasi-isomorphism.*

Proof. The proof is similar to that of Prop.A.1.1. We use Lem.A.1.3 to prove $C_{PL}(-; \mathcal{L})$ is corepresentable. \square

We can define an adjoint pair

$$T_{PL} : \text{Set}^{\square} \rightleftarrows (\text{dgCat}^{\text{cl}})^{\text{op}} : \langle - \rangle.$$

We can see this is a Quillen pair, using Lem.A.1.3 and Prop.A.1.4. This fact is implicitly used in the proof of Lem.4.1.5.

A.2 Tannakian theory

In this subsection, we summarize Tannakian theory of [11]. It states a duality between affine group schemes and certain closed k -categories. The category corresponding to a group scheme is the category of finite dimensional representations of it. We give a characterization of such categories and describe how the group scheme is recovered.

Let Vect' be the category of all finite dimensional k -vector spaces and linear maps.

Definition A.2.1. *A closed k -category $T \in \text{Cat}^{\text{cl}}$ (see Def.2.1.1) is said to be a neutral Tannakian category if it satisfies the following conditions.*

1. T is an Abelian category.
2. $\text{Hom}_T(\mathbf{1}, \mathbf{1}) \cong k$.
3. There exists a morphism $\omega : T \rightarrow \text{Vect}' \in \text{Cat}^{\text{cl}}$ which is faithful and exact.

In the above definition, we use Vect' instead of Vect in order to make neutral Tannakian categories stable under equivalences of Cat^{cl} . (Recall that Vect is a small full subcategory of Vect' such that the natural inclusion $\text{Vect} \rightarrow \text{Vect}'$ is an equivalence, see the paragraph above Def.2.1.3.)

Example A.2.2. *Let Γ (resp. G) be a discrete group (resp. an affine group scheme over k). The closed k -category of finite dimensional k -representations of Γ (resp. G) $\text{Rep}(\Gamma)$ (resp. $\text{Rep}(G)$) is a neutral Tannakian category.*

Let $k - \text{Alg}$ denote the category of commutative and unital k -algebras and Grp denote the category of groups.

Example A.2.3. *Let $\underline{G} : k - \text{Alg} \rightarrow \text{Grp}$ be a functor. A finite dimensional representation of \underline{G} is a pair of a vector space $V \in \text{Vect}$ and a natural transformation $r_- : \underline{G} \Rightarrow \underline{\text{GL}}(V) : k - \text{Alg} \rightarrow \text{Grp}$, where $\underline{\text{GL}}(V)$ is the functor given by $k - \text{Alg} \ni R \mapsto \text{GL}_R(V \otimes_k R) \in \text{Grp}$. There is an obvious notion of morphisms between finite dimensional representations of \underline{G} . We denote the category of finite dimensional representations of \underline{G} by $\text{Rep}(\underline{G})$ and the forgetful functor $\text{Rep}(\underline{G}) \rightarrow \text{Vect}$ by $\omega_{\underline{G}}$. $\text{Rep}(\underline{G})$ has an closed tensor structure such that $\omega_{\underline{G}}$ is a morphism of closed k -categories. Then $\text{Rep}(\underline{G})$ is a neutral Tannakian category. If \underline{G} is represented by an affine group scheme G , $\text{Rep}(\underline{G})$ is equivalent to the category $\text{Rep}(G)$.*

For a neutral Tannakian category T and a morphism $\omega : T \rightarrow \text{Vect}' \in \text{Cat}^{\text{cl}}$ which is exact and faithful, we define a functor $\underline{\text{Aut}}^{\otimes}(\omega) : k - \text{Alg} \rightarrow \text{Grp}$ by

$$k - \text{Alg} \ni R \mapsto \text{Aut}^{\otimes}(\omega \otimes_k R) \in \text{Grp}.$$

Here $\text{Aut}^\otimes(\omega \otimes_k R)$ is the group of tensor preserving natural isomorphisms $\alpha : \omega \otimes_k R \Rightarrow \omega \otimes_k R : T \rightarrow R - \text{Mod}$ ($\omega \otimes_k R : T \rightarrow R - \text{Mod}$ is the morphism from T to the closed k -category of R -modules defined by $T \ni t \mapsto \omega(t) \otimes_k R \in R - \text{Mod}$). We define a morphism of closed k -categories $\tilde{\omega} : T \rightarrow \text{Rep}(\underline{\text{Aut}}^\otimes(\omega))$ by

$$T \ni t \mapsto (\omega(t), ev(t) : \underline{\text{Aut}}^\otimes(\omega) \xrightarrow{\cong} \underline{\text{GL}}(\omega(t))) \in \text{Rep}(\text{Aut}^\otimes(\omega)),$$

where $ev(t)$ is the evaluation of elements of $\text{Aut}^\otimes(\omega)$ at t .

Theorem A.2.4 (Thm.2.11 of [11]). *We use the above notations.*

- (1) *The functor $\underline{\text{Aut}}^\otimes(\omega)$ is represented by an affine group scheme, which is called the Tannakian dual of (T, ω) .*
- (2) *The above morphism $\tilde{\omega} : T \rightarrow \text{Rep}(\underline{\text{Aut}}^\otimes(\omega))$ is an equivalence of categories.*
- (3) *Suppose $(T, \omega) = (\text{Rep}(G), \omega_G)$ for some affine group scheme G . Let \underline{G} be the functor corresponding to G . There is a natural isomorphism $\underline{G} \cong \underline{\text{Aut}}^\otimes(\omega)$ defined by $\underline{G}(R) \ni g \mapsto g \cdot \in \text{Aut}^\otimes(\omega_G \otimes_k R)$.*

Corollary A.2.5. *The pro-algebraic completion Γ^{alg} of a discrete group Γ is isomorphic to the Tannakian dual of $\text{Rep}(\Gamma)$ which is equipped with the natural "evaluation" $\Gamma \rightarrow \text{Aut}^\otimes(\omega) = \underline{\text{Aut}}^\otimes(\omega)(k)$ given by*

$$\Gamma \ni \gamma \mapsto [\text{Rep}(\Gamma) \ni (V, r) \mapsto (r(\gamma) : V \rightarrow V)].$$

The maximal reductive quotient G^{red} of an affine group scheme G is isomorphic to the Tannakian dual of the full subcategory of $\text{Rep}(G)$ consisting of semi-simple representations. (This is closed under tensors and internal homs.)

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